

# COMPUTING MIXED VOLUME AND ALL MIXED CELLS IN QUERMASINTEGRAL TIME

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**ABSTRACT.** The mixed volume counts the roots of generic sparse polynomial systems. Mixed cells are used to provide starting systems for homotopy algorithms that can find all those roots, and track no unnecessary path. Up to now, algorithms for that task were of enumerative type, with no general non-exponential complexity bound. A geometric algorithm is introduced in this paper. Its complexity is bounded in the average and probability-one settings in terms of some geometric invariants: quermassintegrals associated to the tuple of convex hulls of the support of each polynomial. Besides the complexity bounds, numerical results are reported. Those are consistent with an output-sensitive running time for each benchmark family where data is available. For some of those families, an asymptotic running time gain over the best code available at this time was noticed.

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## 1. INTRODUCTION

The *mixed volume* of an  $n$ -tuple of convex bodies  $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ ,  $\mathcal{A}_i \subset \mathbb{R}^n$  is defined by

$$V(\mathcal{A}_1, \dots, \mathcal{A}_n) \stackrel{\text{def}}{=} \frac{1}{n!} \frac{\partial^n}{\partial t_1 \partial t_2 \dots \partial t_n} \text{Vol}(t_1 \mathcal{A}_1 + \dots + t_n \mathcal{A}_n)$$

where  $t_1, \dots, t_n \geq 0$  and the derivative is computed at  $t = 0$ . It generalizes ordinary volume:

$$V(\mathcal{A}, \mathcal{A}, \dots, \mathcal{A}) = \text{Vol}(\mathcal{A}).$$

Mixed volume was introduced by Minkowski (1901) in connection with the *quermassintegrals*  $V(\mathcal{A}, \mathcal{A}, B^3)$  and  $V(\mathcal{A}, B^3, B^3)$ , where  $B^3$  stands for the unit 3-ball. Those quermassintegrals are equal (up to a factor) to the area and the total mean curvature of  $\partial\mathcal{A}$ .

In this paper,  $A_1, \dots, A_n$  are finite subsets of  $\mathbb{Z}^n$ . We will provide an algorithm to compute the scaled mixed volume

$$V = n! V(\text{Conv}(A_1), \dots, \text{Conv}(A_n))$$

together with a set of lower mixed facets for a random lifting (in modern language, a zero-dimensional tropical variety). The BKK bound (Bernstein, 1975; Bernstein et al., 1976) states that  $V$  is the number of roots in  $(\mathbb{C}^\times)^n$  of a generic system of Laurent polynomials  $f_1(\mathbf{x}) = \dots = f_n(\mathbf{x}) = 0$  where

$$(1) \quad f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} f_{ia} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}, \quad 1 \leq i \leq n$$

where the  $f_{ia}$  are complex numbers. Huber and Sturmfels (1995) suggested to use the lower mixed facets (Def.12) to produce a starting system for homotopy algorithms to solve sparse polynomial equations  $f_1(\mathbf{x}) = \dots = f_n(\mathbf{x}) = 0$  with  $f_i$  as above. Emiris and Canny (1995) introduced the first incremental algorithms for computing mixed volume and mixed cells. For a certain time, computing the starting system was a bottleneck for polyhedral homotopy based polynomial solving software (Lee and Li, 2011, p.98). Later breakthroughs by Gao and Li (2000), Li and Li (2001), Gao et al. (2005), Mizutani et al. (2007), Lee and Li (2011), and Chen et al. (2014) provided efficient practical implementations through enumerative algorithms (Remark 14). The complexity properties of those algorithms are not well understood.

The algorithm ALLMIXEDCELLS in page 21 is geometric in nature. This will allow for a complexity bound in terms of geometric invariants (quermassintegrals).

Before writing an algorithm or stating complexity results, one should fix a model of computation. In this paper, an *algorithm* is a randomized *real Random Access Machine* (real RAM) (Preparata and Shamos, 1985). Arithmetic operations  $+$ ,  $-$ ,  $\times$ ,  $/$ ,  $\sqrt{\phantom{x}}$ ,  $\log(\phantom{x})$ ,  $\sin(\phantom{x})$ ,  $\cos(\phantom{x})$  are allowed and cost one unit of time. Memory access is also assumed to be performed at unit cost. In addition, a *randomized* real RAM has access to an unlimited supply of independently uniformly distributed random numbers in  $[0, 1]$ . The running time of a machine with a fixed input is therefore a random variable. Henceforth, the expressions *with probability one* and *on average* refer to the product measure of  $[0, 1]^\infty$ .

Let  $\mathcal{A}_i = \text{Conv}(A_i)$ ,  $\mathcal{A} = \mathcal{A}_1 + \dots + \mathcal{A}_n$  and let  $B^n$  be the unit  $n$ -ball of radius 1. Let  $d_i = \dim \text{Conv}(A_i)$ . Let  $V_i = d_i! \text{Vol}(\text{Conv}(A_i))_{d_i}$  be the generic root bound of an unmixed polynomial system of support  $A_i$ . Let  $0 \leq E_i < \#A_i$  be the numbers to be formally defined in section 5 (but see Remark 6 below).

**Theorem 1.** *With probability one, the algorithm ALLMIXEDCELLSFULL stated on page 23 produces all the lower mixed facets in time bounded by  $O(T + T')$  arithmetic operations, where*

$$(2) \quad T = \left( \sum_{i=2}^n v_i \right) \left( n^2 \sum_{i=1}^n E_i + \log \sum_{i=2}^n v_i \right),$$

$$(3) \quad T' = (\max V_i) \left( n^2 \sum_{i=1}^n \#A_i + \log \max_{i=1, \dots, n} V_i \right),$$

and  $v_i$  is a random variable satisfying the two bounds below:

(a) *With probability one,*

$$v_i \leq n! V(\mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}, B^n, \dots, B^n).$$

(b) *Let  $\bar{v}_i$  be the average of  $v_i$ , then*

$$\bar{v}_i \leq \frac{n!}{2^{n-i}} V(\mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}, B^n, \dots, B^n).$$

*Remark 2.* If the polytopes  $A_i$  are represented by dense  $\#A_i \times n$  matrices, then  $n \sum E_i$  is a lower bound for the input size  $S$ . So the complexity can be bounded above by

$$(4) \quad O(W(nS + \log W)), \quad W = \max \left( \max_i (V_i), \sum_i v_i \right).$$

*Remark 3.* Because of monotonicity of the mixed volume,  $v_i \leq n! V(\mathcal{A}, \dots, \mathcal{A}, B^n, \dots, B^n)$ . Also,  $V_i \leq n! \text{Vol}(\mathcal{A})_n$ . If  $\mathcal{A}$  contains a copy of the unit ball, then  $v_i \leq n! \text{Vol}(\mathcal{A})$ .

*Remark 4.* In the probability-one bound (a) for  $v_i$ , one can replace  $B^n$  by  $\beta_n = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$ , the  $n$ -orthoplex (Sec. 6). This replacement gives an exponentially smaller bound when  $i$  is small. Assuming that  $\text{Vol}(\mathcal{A}) \neq 0$ , we obtain

$$v_i \leq n! 2^{n-i} \text{Vol}(\mathcal{A}).$$

It is not clear whether a similar bound holds for  $\bar{v}_i$  in the average case analysis (b).

*Remark 5.* Assume that  $\dim \text{Conv}(A_i) = n$ . Let  $\delta_i$  denote the radius of the inscribed sphere to  $\mathcal{A}_i$  and  $\Delta$  the radius of the circumscribed sphere to  $\mathcal{A}$ . Then,

$$v_i \leq n! \frac{\Delta}{\delta_i \delta_{i+1} \dots \delta_n} V(\mathcal{A}_1, \dots, \mathcal{A}_n) \quad \text{and} \quad \bar{v}_i \leq \frac{n!}{2^{n-i}} \frac{\Delta}{\delta_i \delta_{i+1} \dots \delta_n} V(\mathcal{A}_1, \dots, \mathcal{A}_n).$$

*Remark 6.* The bound  $T'$  represents the cost of computing a lower convex hull of a random lifting for each of the polytopes  $A_1, \dots, A_n$ . Typically  $T' \ll T$  but counterexamples may be produced. In a previous version of this paper, the algorithm was assumed to receive those lower convex hulls as precomputed information.  $E_i$  is the degree of the 1-skeleton of the lower convex hull for the lifting of  $A_i$  (Sec. 5).

From a complexity standpoint, bounding the cost of mixed volume computation in terms of the mixed volume and similar invariants is the best that we can aim for. The general problem of computing the mixed volume is known to be  $\#P$ -complete. This follows from the famous result by Khachiyan (1989) that computing *volumes* of convex polytopes is already  $\#P$ -hard. Barvinok (1997) suggested approximating mixed volumes by the mixed volume of ellipsoids. Gurvits (2009) obtained an approximation within a factor exponential in the number of variables, and showed that the same ratio could not be obtained with a deterministic algorithm in the Oracle setting. Dyer et al. (1998) provided good approximations in certain special cases, but showed also that computing mixed volumes of zonotopes is already  $\#P$ -hard.

Emiris (1996) was able to bound the complexity of the algorithm by Emiris and Canny (1995) for enumerating mixed cells in terms of the volume of the Minkowski sum of all polytopes. Assuming that all polytopes have non-zero  $n$ -dimensional volume, he deduced bounds for the bit-complexity. Simultaneously, Verschelde et al. (1996) introduced dynamic lifting and also obtained complexity bounds. In both papers the complexity bounds depend on the number of lower facets, not necessarily mixed facets. More recently Emiris and Vidunas (2014) gave specific formulas for certain semi-mixed volumes. Emiris and Fisikopoulos (2014) devised an algorithm to compute the mixed volume without actually computing the mixed cells.

The algorithm in this paper visits  $v_n$  lower facets including all the mixed cells. Those facets are ‘dual’ to a certain tropical curve, not necessarily connected. The precise definition of this tropical curve requires the introduction of a mixed Legendre transform, which allows to efficiently represent tropical varieties as specific subsets of the viable set of some linear programming problem. The precise formalism is introduced in Section 2.

In Section 3, it is proved that each connected component of this tropical curve cuts a generic affine hyperplane with probability 1. This allows to find all the connected components by a dimensional induction. Each component of the tropical curve is explored by a particular pivoting procedure, that takes into account the structure of the problem. Those procedures are explained in Section 4, together with the procedures for pivoting from one induction level to the other. Because of numerical stability reasons, the generic affine hyperplane is sent to infinity and the pivoting procedures use nonstandard real numbers (real polynomials in a parameter  $R \rightarrow \infty$ ).

The algorithm and intermediate complexity bounds are given in Section 5. An important complexity gain is obtained by assigning a *hash value* to every lower face. This allows to efficiently store sets of explored and unexplored lower faces as a balanced tree. The proof of Theorem 1 is completed in Sections 6 and 7, where the numbers  $v_d$  and  $\bar{v}_d$  get bounded in terms of mixed volumes (quermassintegrals).

An actual implementation of the algorithm is described in sections 8 to 10. This is part of a long-term project to produce a toric homotopy based polynomial system solver. The source code is available at <http://sourceforge.net/projects/pss5/> and licensed under GNU Public License. At this time, only the mixed volume section of the code is complete and fully tested. There is a glossary of notations at the end of the paper.

The rationale for including sections 8 to 10 in this paper is to substantiate the following claims:

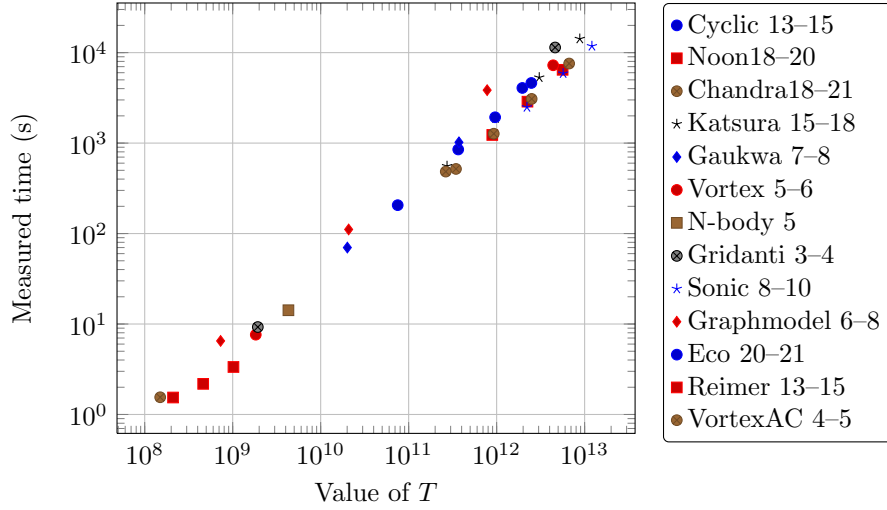


FIGURE 1. Measured running time (using 8 cores) against the invariant  $T$  from equation (2) for several benchmark examples. Data from table 1 page 30.

*Claim 7.* The model of computation is realistic, in the sense that the complexity bound in Theorem 1 accurately describes the running-time measurements for a publicly available implementation of the algorithm.

Experiments were performed on a large number of examples, including some very large benchmark systems (Fig. 1). It is worth to mention that numerical stability issues did arise. Those were circumvented by a careful error analysis and a recovery step in the linear algebra routines (Sec. 8). The experiments in Fig 1 show no noticeable running time increase due to the eventual recovery step.

*Claim 8.* The algorithm is scalable.

Modern computers are built with multiple cores, and serial complexity analysis does not guarantee a competitive parallel running-time. The program was successfully tested on a parallel environment with up to 8 nodes running 8 cores each.

When analyzing parallel algorithms, the most important complexity invariant is the communication complexity. The parallel version of the algorithm will exchange at most  $O(\sum v_d)$  messages of size  $O(n)$ . Again, this bound alone does not imply good practical scalability properties, so experimentation is necessary.

For each given polynomial system, let  $T_N$  be the measured running time with  $N$  cores. In the benchmark families tested, the running time was of the order of  $O(N^{0.93})$ .

Another parallel algorithm for the same problem was described by Chen et al. (2014). Figure 5 in their paper shows the speedup factor for the *Cyclic-15* benchmark example in a similar multi-node environment. From their picture, their speed-up factor  $T_{32}/T_{64}$  from 32 to 64 cores is around 1,80 against 1,96 obtained here.

*Claim 9.* The program performance is comparable to the best available code.

Other free software for mixed volume computations are MixedVol by Gao et al. (2005), DEMICs by Mizutani and Takeda (2008) and PHCpack by Verschelde

(1999). Closed source programs are MixedVol2.0 by Lee and Li (2011) and MixedVol3.0 by Chen et al. (2014).

At this time, Lee and Li (2011) and Chen et al. (2014) have the best published timings for the problem of computing mixed volumes and mixed cells. Since the algorithm in this paper is different, the results obtained here are better for some benchmark families and worse for others.

Overall, the implementation of ALLMIXEDCELLSFULL appeared to be reliable for systems with *output size* of around  $nV \simeq 10^7$  and beyond. In each of the benchmark families tested, the running time grows moderately with respect to the output size. In some of the benchmark families, a big performance gain was obtained by using a random path heuristic.

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## 2. MIXED LEGENDRE TRANSFORM AND TROPICAL VARIETIES

In order to introduce our main tools, it is convenient to work in a more general setting. Some of the supports  $A_i$  may be repeated, and there is some work to save by considering *semi-mixed volumes*, that is mixed volumes with multiplicities. Through this paper,  $A_1, \dots, A_s$  are finite subsets of  $\mathbb{Z}^n$ ,  $s \leq n$ . Multiplicities  $m_1 + \dots + m_s = n$  are fixed,  $m_i \geq 0$ . The semi-mixed volume is defined by

$$\begin{aligned} V &= V(\text{Conv}(A_1), m_1; \dots; \text{Conv}(A_s), m_s) \\ &\stackrel{\text{def}}{=} V(\underbrace{\text{Conv}(A_1), \dots, \text{Conv}(A_1)}_{m_1 \text{ times}}, \dots, \underbrace{\text{Conv}(A_s), \dots, \text{Conv}(A_s)}_{m_s \text{ times}}). \end{aligned}$$

Thus,  $n!V$  is the generic number of roots in  $(\mathbb{C}^\times)^n$  of polynomial systems of the form

$$f_{ij}(\mathbf{x}) = 0, \quad 1 \leq j \leq m_i, \quad 1 \leq i \leq s,$$

with

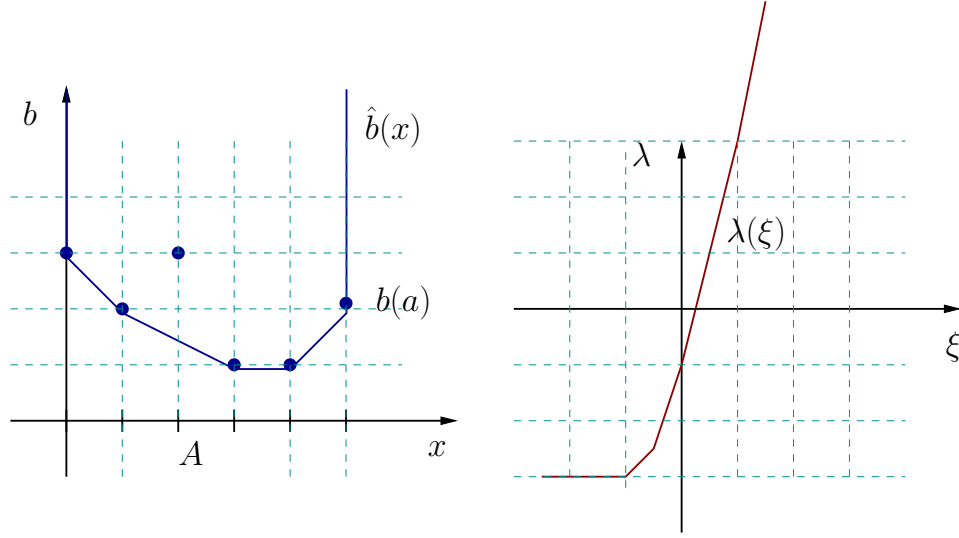
$$f_{ij}(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} f_{ij\mathbf{a}} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}.$$

*Definition 10.* A real function  $b : A_1 \sqcup \dots \sqcup A_s \rightarrow \mathbb{R}$  is in *general position* if and only if, for any subset  $S$  of  $\{[-\mathbf{e}_i, \mathbf{a}, b(i, \mathbf{a})] : \mathbf{a} \in A_i\} \subset \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}$  of cardinality  $k \leq n + s + 1$ , either the vectors in  $S$  are linearly independent or  $[0, \dots, 0, 1]$  is a linear combination of the points in  $S$ .

In particular,  $b$  is in general position with probability 1, that is outside of a certain set of Lebesgue measure zero. The function  $b$  appears in mixed volume computation papers as a *random lifting*.

Let  $b_i = b|_{A_i}$ . The *Legendre dual* of  $b_i$  is the function  $\lambda_i : (\mathbb{R}^n)^* \rightarrow \mathbb{R}$  defined by

$$\lambda_i(\boldsymbol{\xi}) = \max_{\mathbf{a} \in A_i} \boldsymbol{\xi}(\mathbf{a}) - b(\mathbf{a}).$$

FIGURE 2. Legendre duality for  $s = 1$ .

The function  $\lambda_i$  is convex. Its Legendre dual is the lower convex hull of  $b_i$ , defined as the largest convex function  $\hat{b}_i : \text{Conv}(A_i) \rightarrow \mathbb{R}$  with  $\hat{b}_i(\mathbf{a}) \leq b_i(\mathbf{a})$  for all  $\mathbf{a} \in A_i$ . (Fig.2). Its *epigraph*  $\{(\mathbf{x}, y) : y \geq \hat{b}(\mathbf{x})\}$  can be seen as the convex hull of the set  $\{(\mathbf{a}, b_i(\mathbf{a})) : \mathbf{a} \in A_i\}$  and a point *at infinity*  $(0, \infty)$ . The non-vertical faces of the epigraph project onto a subdivision of the Newton polytope  $\text{Conv}(A_i)$ .

*Remark 11.* In the language of tropical algebraic geometry, the Legendre dual  $\lambda_i$  of  $b_i$  is a tropical polynomial.

Assume that  $b$  is in general position. To any  $\boldsymbol{\xi} \in (\mathbb{R}^n)^*$ , we associate the numbers  $m_1(\boldsymbol{\xi}), \dots, m_s(\boldsymbol{\xi})$  such that  $\lambda_i(\boldsymbol{\xi})$  is attained for exactly  $m_i(\boldsymbol{\xi}) + 1$  values of  $\mathbf{a} \in A_i$ . We also associate to the pair  $(i, \boldsymbol{\xi})$  a face  $L_{i, \boldsymbol{\xi}}$  of  $\text{Graph}(\hat{b}_i)$ ,

$$(5) \quad L_{i, \boldsymbol{\xi}} = \left\{ (\mathbf{x}, \hat{b}(\mathbf{x})) : \mathbf{x} \in \text{Conv}(A_i) \text{ and } \lambda_i(\boldsymbol{\xi}) = \boldsymbol{\xi}(\mathbf{x}) - \hat{b}(\mathbf{x}) \right\}.$$

Let  $t_1, \dots, t_s > 0$  be indeterminates. We consider now the mixed lifting  $t_1 b_1 + \dots + t_s b_s$  of the set of formal linear combinations  $t_1 A_1 + \dots + t_s A_s$  by:

$$t_1 \mathbf{a}_1 + \dots + t_s \mathbf{a}_s \mapsto t_1 b_1(\mathbf{a}_1) + \dots + t_s b_s(\mathbf{a}_s).$$

No ordering between the  $t_i$  is assumed. Yet, to every  $\boldsymbol{\xi} \in \mathbb{R}^n$ , we can associate a face  $L_{\boldsymbol{\xi}}$  of  $\text{Graph}(t_1 \hat{b}_1 + \dots + t_s \hat{b}_s)$ ,

$$(6) \quad L_{\boldsymbol{\xi}} = t_1 L_{1, \boldsymbol{\xi}} + \dots + t_s L_{s, \boldsymbol{\xi}}.$$

The face  $L_{\boldsymbol{\xi}}$  is well defined because the  $L_{i, \boldsymbol{\xi}}$  are independent with respect to the specialization of the variables  $t_i$ . This face is said to be of type  $(m_1(\boldsymbol{\xi}), \dots, m_s(\boldsymbol{\xi}))$ . Since  $b$  is in general position, the dimension of  $L_{\boldsymbol{\xi}}$  is  $m_1(\boldsymbol{\xi}) + \dots + m_s(\boldsymbol{\xi})$ .

Reciprocally, any  $n$ -dimensional lower facet  $L$  admits a unique vector  $\xi \in (\mathbb{R}^n)^*$  with  $L = L_\xi$ . More generally, let  $L$  be a lower face of any dimension and define  $\Xi(L) = \{\xi : L = L_\xi\}$ . Then  $\Xi(L)$  is a (possibly unbounded) polyhedron of dimension  $n - \sum m_i(\xi)$ , ‘dual’ to  $L$ .

*Definition 12.* Let  $m_1, \dots, m_s$  be the fixed multiplicities associated to each polytope,  $m_1 + \dots + m_s = n$ . A *lower mixed facet* is a lower face of type  $(m_1, \dots, m_s)$ . A *mixed cell* is the projection of a lower mixed facet into  $\mathbb{R}^n$ .

The mixed volume  $V$  is equal to the sum of the volume of the mixed cells. The basic idea for enumerating the mixed facets is to explore certain tropical varieties. One can specify a tropical variety in  $\mathbb{R}^n$  by bounding the  $m_i(\xi)$  below. For instance,

$$T_i \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^n : m_i(\xi) \geq 1\} = \{\xi \in \mathbb{R}^n : \max(\mathbf{a}\xi - b_i(\mathbf{a})) \text{ attained twice}\}$$

is the hypersurface defined by the tropical polynomial

$$\sum_{\mathbf{a} \in A_i} (-b_i(\mathbf{a})) \xi_1^{a_1} \xi_2^{a_2} \cdots \xi_n^{a_n}.$$

Since the lifting  $b_i$  is assumed to be in general position, the set

$$\{\xi \in \mathbb{R}^n : m_i(\xi) \geq 2\}$$

is a codimension 2 hypersurface, which is the stable intersection  $2T_i \stackrel{\text{def}}{=} T_i \cap_{\text{st}} T_i$  (MacLagan and Sturmfels, 2015, Ch.3). More generally if  $1 \leq m \leq n$ ,  $\{\xi \in \mathbb{R}^n : m_i(\xi) \geq m\}$  is the stable intersection  $mT_i = T_i \cap_{\text{st}} \cdots \cap_{\text{st}} T_i$ . The lower mixed facets are the  $L_\xi$  where  $\xi \in X_n$  and  $X_n$  is the point configuration

$$X_n \stackrel{\text{def}}{=} m_1 T_1 \cap_{\text{st}} \cdots \cap_{\text{st}} m_s T_s = \{\xi \in \mathbb{R}^n : m_1(\xi) \geq m_1, \dots, m_s(\xi) \geq m_s\}.$$

In order to find  $X_n$  we will proceed by induction on the dimension  $0 \leq d \leq n$ . At each step we will explore a one-dimensional tropical variety  $G_d$  containing  $X_d$ . Those varieties need to satisfy certain genericity hypotheses, so we proceed as follows:

Let  $F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{R}^n$  be a flag of generic *affine* subspaces. At dimension  $d$ , we will produce  $X_d \subset F_d$  corresponding to certain multiplicities  $m_1^{(d)}, \dots, m_s^{(d)}$ . If  $d < n$ , we will then explore  $G_d$  and produce  $X_{d+1}$ . For  $i \in \{1, \dots, s\}$  and  $d \in \{0, \dots, n\}$ , we choose the multiplicities  $0 = m_i^{(0)} \leq m_i^{(1)} \leq \cdots \leq m_i^{(n)} = m_i \in \mathbb{N}_0$  so that  $\sum_i m_i^{(d)} = d$ . To do this, we start at  $d = 0$  with all the  $m_i^{(0)} = 0$  and then increase exactly one of the  $m_i^{(d)}$  at each step  $1 \leq d \leq n$ . We define:

$$\begin{aligned} X_d &= F_d \cap \left( m_1^{(d)} T_1 \cap_{\text{st}} \cdots \cap_{\text{st}} m_s^{(d)} T_s \right) \\ G_d &= F_d \cap \left( m_1^{(d-1)} T_1 \cap_{\text{st}} \cdots \cap_{\text{st}} m_s^{(d-1)} T_s \right). \end{aligned}$$

Using explicit notation,

$$\begin{aligned} X_d &= \{\xi \in F_d : \forall i, m_i(\xi) \geq m_i^{(d)}\} \\ G_d &= \{\xi \in F_d : \forall i, m_i(\xi) \geq m_i^{(d-1)}\}. \end{aligned}$$

The induction starts with  $X_0 = \{F_0\}$ . The induction step is possible because of the result below, stated in classical terms:



*Theorem 13.* Assume that  $V(\text{Conv}(A_1), m_1; \dots; \text{Conv}(A_s), m_s) \geq 1$ . If  $b : A_1 \sqcup \dots \sqcup A_n \rightarrow \mathbb{R}$  is in general position and the flag  $F_d$  generic, then

- (a) Each set  $G_d$ ,  $d = 1, \dots, n$ , is a finite closed union of line segments and half-lines.
- (b)  $X_{d-1} = G_d \cap F_{d-1}$ .
- (c) Each connected component of  $G_d$  intersects  $F_{d-1}$  at least in one point.
- (d) (Transversality) All points in  $G_d \cap F_{d-1}$  are in the interior of a line segment or a half-line of  $G_d$ .

Moreover, a certain balancing condition (Lemma 16) holds for the edges incident to a vertex.

The *structure theorem* of Tropical Algebraic Geometry (Cartwright and Payne, 2012; Maclagan and Sturmfels, 2015) guarantees the connectedness of one-dimensional intersections of tropical varieties  $G = G_d$ , as long as the underlying complex variety is irreducible. Example 15 shows how  $G_n$  can fail to be connected on a positive measure set of liftings  $b$  when the underlying complex variety is reducible. The precise combinatorial conditions for the connectedness of tropical hypersurfaces are described by Yu (2015).

The main tool for proving Theorem 13 is the mixed Legendre transform  $\xi \mapsto \lambda_1(\xi), \dots, \lambda_s(\xi)$ . As before, the  $t_1, \dots, t_s$  are positive indeterminates. The ‘epigraph’ of the function  $t_1 b_1 + \dots + t_s b_s$  is the set of all the pairs  $(\xi, t_1 \lambda_1 + \dots + t_s \lambda_s)$  so that  $(\lambda, \xi)$  is a solution of the system of inequalities

$$(7) \quad \mathbf{C} \begin{bmatrix} \lambda \\ \xi \end{bmatrix} \leq \mathbf{b}$$

where  $\mathbf{C}$  is the *Cayley matrix*

$$(8) \quad \mathbf{C} = \begin{bmatrix} -1 & 0 & \dots & 0 & & \\ \vdots & \vdots & & \vdots & & \\ -1 & 0 & \dots & 0 & & \\ 0 & -1 & \dots & 0 & & \\ \vdots & \vdots & & \vdots & & \\ 0 & -1 & \dots & 0 & & \\ & & \ddots & & & \\ & & & \ddots & & \\ 0 & 0 & \dots & -1 & & \\ \vdots & \vdots & & \vdots & & \\ 0 & 0 & \dots & -1 & & \end{bmatrix} \begin{matrix} A_1 \\ \\ A_2 \\ \\ \vdots \\ \\ A_s \end{matrix}$$

and each support  $A_i$  is represented by a matrix with rows  $\mathbf{a} \in A_i$ . As no confusion can arise, we use the same symbol for the support and its representing matrix.

A lower face  $L$  of dimension  $d$  can be represented by a pair  $(\lambda(\xi), \xi)$  so that  $L = L(\xi)$ , with exactly  $s+d$  equalities in (7). In the language of linear programming, those are known as the *active constraints* while the strict inequations are deemed *inactive*.

The same lower face  $L$  can also be represented by its set of active constraints. Its dual  $\{\xi\} = \Xi(L)$  is the set of solutions of

$$\begin{aligned} [-e_i, \mathbf{a}] \begin{bmatrix} \lambda(\xi) \\ \xi \end{bmatrix} &= b(i, \mathbf{a}) && \text{for } (i, \mathbf{a}) \text{ an active constraint,} \\ [-e_i, \mathbf{a}] \begin{bmatrix} \lambda(\xi) \\ \xi \end{bmatrix} &< b(i, \mathbf{a}) && \text{for } (i, \mathbf{a}) \text{ inactive.} \end{aligned}$$

The tropical varieties  $G_d$  can be explored by *pivoting* from each  $d$ -dimensional lower face to its neighboring faces. The numerics for pivoting are a fallback from the techniques of the simplex algorithm. This is explained in section 4.

*Remark 14.* The other known practically efficient algorithms are those by Mizutani et al. (2007) and (Lee and Li, 2011). Both algorithms function by enumerating certain viable lower faces of dimension  $k$ , for  $0 \leq k \leq n$ . To simplify the discussion we assume  $n = s$  and  $m_1 = \dots = m_n = 1$ . In the language of this paper, those algorithms enumerate lower faces corresponding to solutions of (7) with exactly 2 equalities for the  $i$ -th block of  $\mathbf{C}$ , only for  $i$  in a cardinality  $k$  subset of  $\{1, \dots, n\}$ . Given one of such faces for  $k < n$ , they extend it to a  $k+1$  face using ideas of linear programming. It can happen that some faces are not extensible, and a lot of effort is made to devise heuristics that prune the decision tree as early as possible. No complexity analysis is available.

*Example 15.* (Fig.3). Consider the *Cyclic-3* polynomial system, and replace the coefficients with random coefficients:

$$\begin{aligned} c_0x_1 + c_1x_2 + c_2x_3 &= 0 \\ c_3x_1x_2 + c_4x_2x_3 + c_5x_3x_1 &= 0 \\ c_6x_1x_2x_3 + c_7 &= 0 \end{aligned}$$

The zero set of the first two equations is reducible: eliminate  $x_3$  from the first equation and substitute in the second to obtain an equation of the form  $Ax_1^2 + 2Bx_1x_2 + Cx_2^2$ , which clearly factors. Let  $m_1^{(1)} = m_2^{(2)} = m_3^{(3)} = 1$  as in the figure.

The figure show that for the following values of  $b_i = c_i$ , the tropical variety  $G_3$  is disconnected for a generic flag:

$b_0 = 0.0681718062929322$	$b_3 = 0.8654168322306781$	$b_6 = 0.6575801418616753$
$b_1 = 0.2764482146232536$	$b_4 = 0.6630347993316177$	$b_7 = 0.2139433513437121$
$b_2 = 0.4266688073141105$	$b_5 = 0.2369372029023467$	

The mixed vertices of  $X_3$  are:

$$X_3 = \left\{ \begin{bmatrix} 1.7041246197535198 \dots 10^{-01} \\ -2.5568513445391900 \dots 10^{-1} \\ 5.2890946299653028 \dots 10^{-01} \end{bmatrix}, \begin{bmatrix} 2.8794667050532442 \dots 10^{-1} \\ 4.9622307883564581 \dots 10^{-1} \\ -3.4053295882300698 \dots 10^{-1} \end{bmatrix} \right\}$$

### 3. PROOF OF THEOREM 13

A *vertex* in  $G_d$  is a point  $\xi \in F_d$  such that  $\sum m_i(\xi) = d$ . An *edge* in  $G_d$  is a one-dimensional intersection  $F_d \cap L_\xi$  where  $L_\xi$  is the lower face associated to a vector  $\xi$  as in (6). Equivalently, an edge is the projection onto  $\xi$ -space of a non-empty solution set of (7) in  $F_d$  with prescribed  $m_1^{(d-1)} + 1, \dots, m_s^{(d-1)} + 1$  equalities in the respective block, and no more equalities.

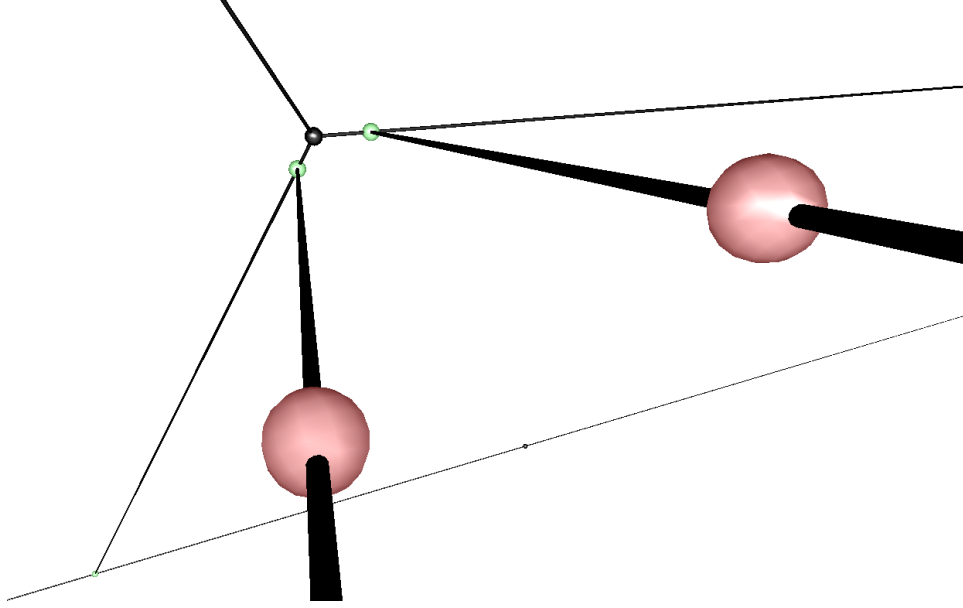


FIGURE 3. Cyclic-3 polynomial system:  $X_3$  is represented by the two big balls,  $G_3$  by the two thick lines going through  $X_3$ .  $G_3$  cuts  $F_2$  on  $X_2$  (light green balls).  $G_2$  is visible as three half-lines starting at the black ball afar. Albeit at infinity,  $F_2$  was artificially represented as a usual plane. This is why the lines in  $G_3$  ‘stop’ at  $F_2$ . Similarly,  $F_1$  is represented as a usual line, and parts of  $G_1$  are visible as the thin line from bottom left to right. This figure was obtained for  $m_1^{(1)} = m_2^{(2)} = m_3^{(3)} = 1$ .

The set  $G_d$  is a finite union of vertices and edges. Edges may be bounded or unbounded. Bounded edges are open segments, whose end points are vertices of  $G_d$ . Unbounded edges cannot be a line, for otherwise all the  $A_i$ ’s would be contained in an hyperplane orthogonal to that line. Therefore, unbounded edges are half-lines, bounded in one side by a vertex of  $X_0$ . (Fig.4) Recall that  $m_i(\xi)$  is the number of values of  $\mathbf{a} \in A_i$  such that  $\lambda_i(\xi) = \mathbf{a}\xi - b(i, \mathbf{a})$ .

*Lemma 16.* Let  $\xi$  be a vertex of  $G_d$  and let  $1 \leq q \leq s$  be the unique integer with  $m_q(\xi) = m_q^{(d-1)} + 1$ . Then, there are precisely  $m_q(\xi) + 1 = m_q^{(d-1)} + 2$  edges of  $G_d$  with endpoint  $\xi$ . Moreover, those edges are of the form  $\{\xi + t\Delta_j\xi : t \in (0, I_j)\}$  where  $I_j$  is either a strictly positive real number, or infinity. The following balancing condition holds:

$$(9) \quad \sum \Delta_j \xi = 0.$$

This follows directly from the *balancing condition* in tropical algebraic geometry. For the benefit of general readers, a self-contained proof is given below.

*Proof.* Let  $\mathbf{C}_{\text{act}}$ ,  $\mathbf{b}_{\text{act}}$  be the submatrix of the Cayley matrix (resp. subvector of  $\mathbf{b}$ ) with rows in the set of active constraints of vertex  $\xi$ , plus the  $n - d$  affine constraints in  $\xi$  that define  $F_d$ . So  $\mathbf{C}_{\text{act}}$  is a  $(s+n) \times (s+n)$  matrix and  $\mathbf{b}_{\text{act}} \in \mathbb{R}^{s+n}$ . Let  $\mathbf{C}_{\text{inact}}$

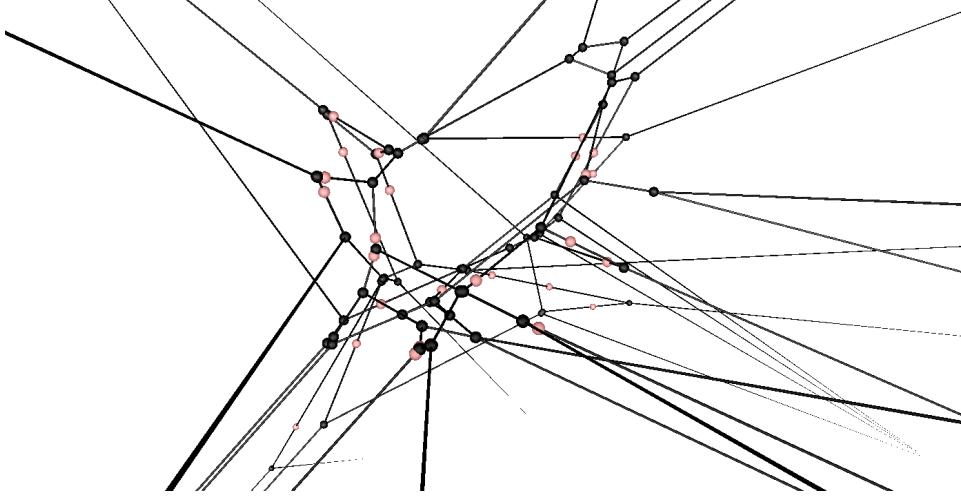


FIGURE 4. The Cohn 3 polynomial system from Posso.  $X_4$  is pictured as light balls and the rest of  $G_4$  as dark balls and lines.

(resp.  $\mathbf{b}_{\text{inact}}$ ) be the submatrix (resp. subvector) of inactive constraints associated to vertex  $\xi$ . Therefore we assumed that

$$\mathbf{C}_{\text{act}} \begin{bmatrix} \lambda \\ \xi \end{bmatrix} = \mathbf{b}_{\text{act}}, \quad \mathbf{C}_{\text{inact}} \begin{bmatrix} \lambda \\ \xi \end{bmatrix} < \mathbf{b}_{\text{inact}}.$$

The edges incident to  $\xi$  are obtained by releasing one of the  $m_q(\xi) + 1$  equalities in the  $q$ -th block.

Suppose we release the  $j$ -th equality, where the index  $j$  corresponds to the row in  $\mathbf{C}_{\text{act}}$  associated to that equality. In particular  $j \in J_q = \{j : 1 \leq j - (\sum_{i < q} m_i^{(d-1)}) - q \leq m_i^{(d-1)} + 2\}$ . We obtain a system of equalities of the form

$$\mathbf{C}_{\text{act}} \begin{bmatrix} \lambda + t\Delta_j \lambda \\ \xi + t\Delta_j \xi \end{bmatrix} = \mathbf{b}_{\text{act}} - te_j$$

where  $e_j$  is the  $j$ -th canonical basis vector and  $t > 0$  is indeterminate. This simplifies to

$$\mathbf{C}_{\text{act}} \begin{bmatrix} \Delta_j \lambda \\ \Delta_j \xi \end{bmatrix} = -e_j.$$

We can solve and find

$$\begin{bmatrix} \Delta_j \lambda \\ \Delta_j \xi \end{bmatrix} = -\mathbf{C}_{\text{act}}^{-1} e_j.$$

The balancing condition follows from the fact that

$$\mathbf{C}_{\text{act}} \begin{bmatrix} e_q \\ 0 \end{bmatrix} = - \sum_{j \in J_q} e_j.$$

Multiplying by  $\mathbf{C}_{\text{act}}^{-1}$ ,

$$\begin{bmatrix} e_q \\ 0 \end{bmatrix} = - \sum_j \mathbf{C}_{\text{act}}^{-1} e_j = \begin{bmatrix} \sum_{j \in J_q} \Delta_j \lambda \\ \sum_{j \in J_q} \Delta_j \xi \end{bmatrix}.$$

From this we deduce that  $\sum_{j \in J_q} \Delta_j \xi = 0$ . □

An immediate consequence of (9) is the following Lemma, to be used in the proof of Theorem 13(c).

*Lemma 17. Let  $\mathbf{Q} \in (\mathbb{R}^n)^*$  be an arbitrary objective function. Then either  $\mathbf{Q} \Delta_j \boldsymbol{\xi} > 0$  for some  $j \in J_q$ , or  $\mathbf{Q} \Delta_j \boldsymbol{\xi} = 0$  for all  $j \in J_q$ .*

In order to pivot from face to face, we need to know the value of  $I_j$  in Lemma 16.

*Lemma 18. In the conditions above,*

$$I_j = \min \left( \{+\infty\} \cup \left\{ t(i, \mathbf{a}) : \mathbf{a} \in A_i, \lambda_i(\boldsymbol{\xi}) < \lambda_i \text{ and } t(i, \mathbf{a}) > 0 \right\} \right)$$

where

$$t(i, \mathbf{a}) = \frac{[-\mathbf{e}_i, \mathbf{a}] \begin{bmatrix} \lambda \\ \boldsymbol{\xi} \end{bmatrix} - b(i, \mathbf{a})}{[-\mathbf{e}_i, \mathbf{a}] \mathbf{C}_{\text{act}}^{-1} \mathbf{e}_j}.$$

*Proof.* In order to find  $I_j$ , we solve

$$\mathbf{C}_{\text{inact}} \begin{bmatrix} \lambda + t \Delta_j \lambda \\ \boldsymbol{\xi} + t \Delta_j \boldsymbol{\xi} \end{bmatrix} \leq \mathbf{b}_{\text{inact}}$$

with exactly one equality. This is the same as

$$\mathbf{C}_{\text{inact}} \begin{bmatrix} \lambda \\ \boldsymbol{\xi} \end{bmatrix} - \mathbf{b}_{\text{inact}} \leq t \mathbf{C}_{\text{inact}} \mathbf{C}_{\text{act}}^{-1} \mathbf{e}_j$$

with  $t > 0$  and exactly one equality. The left hand side is always negative. For each inactive constraint  $[-\mathbf{e}_i, \mathbf{a}]$ ,  $\mathbf{a} \in A_i$ , set

$$t(i, \mathbf{a}) = \frac{[-\mathbf{e}_i, \mathbf{a}] \begin{bmatrix} \lambda \\ \boldsymbol{\xi} \end{bmatrix} - b(i, \mathbf{a})}{[-\mathbf{e}_i, \mathbf{a}] \mathbf{C}_{\text{act}}^{-1} \mathbf{e}_j}.$$

Then  $I_j$  is the minimal positive value of  $t(i, \mathbf{a})$  where  $[-\mathbf{e}_i, \mathbf{a}]$  is an inactive constraint. In case the set of positive values is empty,  $I_j = +\infty$ .  $\square$

*Proof of Theorem 13.* We already checked (a), and (b) holds by construction. We prove (c) now. Let  $\mathbf{Q}_d$  be a non-zero normal vector to  $F_{d-1} \subseteq F_d$  so that  $F_{d-1} = \{\boldsymbol{\xi} \in F_d : \mathbf{Q}_d \boldsymbol{\xi} = r_d\}$ . Each connected component of  $G(d)$  has finitely many vertices. There is a finite number of possible matrices  $\mathbf{C}_{\text{act}}$ . The first  $s + d$  rows of each  $\mathbf{C}_{\text{act}}$  are constraints and the remaining  $n - d$  rows are obtained from constraints  $\mathbb{Q}_{d+1}, \dots, \mathbb{Q}_n$ . Because the flag  $F_0 \subset F_1 \subset \dots \subset F_n$  is generic,  $\mathbf{Q}_d$  is not orthogonal to any of the columns of any  $\mathbf{C}_{\text{act}}^{-1}$ . Thus, at all vertices of  $G(d)$ ,  $\mathbf{Q}_d \Delta_j \boldsymbol{\xi} \neq 0$  and  $\mathbf{Q}_d \Delta_j \boldsymbol{\xi}$  assumes both strictly positive and negative values. As there is a finite number of vertices in each connected component  $C$  of  $G(d)$ , at least one of the strictly positive (resp. strictly negative) values corresponds to a half-line. Hence, the connected component  $C$  has points with  $\mathbf{Q}_d \boldsymbol{\xi} \rightarrow -\infty$  and  $\mathbf{Q}_d \boldsymbol{\xi} \rightarrow +\infty$ .

By the intermediate value theorem,  $C$  must cut  $F_{d-1} = \{\boldsymbol{\xi} : \mathbf{Q}_d \boldsymbol{\xi} = r_d\}$  at least once. This proves (c). The transversality condition (d) follows from the genericity of  $r_d$ .  $\square$

*Remark 19.* If one picks  $r_n \gg r_{n-1} \gg \dots \gg r_{d+1}$  then the transversality condition still holds.

## 4. FACET PIVOTING

In this section we produce the equations for pivoting from a point of  $X_d$  to its neighbors in  $G_d$  (Lemma 23) and also to pivot between  $X_d$  and  $X_{d+1}$  (Lemmas 24 and 25). We start with a well-known Lemma that can be used to relate the matrices of active constraints in two adjacent vertices of  $G_d$ .

*Lemma 20* (Rank 1 updates). *Let  $A$  and  $B$  be  $n \times n$  matrices with  $B = A^{-1}$ . Let  $u, v \in \mathbb{R}^n$ . Then  $A - uv^T$  is invertible if and only if  $v^T B u \neq 0$ . Moreover, if  $A - uv^T$  is invertible,*

$$(A - uv^T)^{-1} = B + \frac{1}{1 - v^T B u} B u v^T B.$$

*Proof.* First of all, notice that  $\det(A - uv^T) = \det(A) \det(I - A^{-1} uv^T) = 1 - v^T A^{-1} u$ . Assuming this is different from zero,

$$(A - uv^T)^{-1} = B(I - uv^T B)^{-1} = B(I + \frac{1}{1 - v^T B u} uv^T B).$$

The last equality follows from multiplying  $I - uv^T B$  and  $I + \frac{1}{1 - v^T B u} uv^T B$ .  $\square$

Now, assume that  $F_d \subset F_{d+1}$  is defined by the equation  $Q_{d+1} \xi = R^{n-d} r_{d+1}$ , where  $r_{d+1} > 0$  and  $R > 0$  is a parameter that will tend to infinity. The reason for the choice  $R \rightarrow \infty$  comes from practical considerations.

Figure 5 shows a typical graph  $G_d$ . The region close to the origin seems overcrowded with interlaced edges, while the ‘spikes’ do not appear to be as numerous as the finite edges. Cutting by a plane at infinity minimizes the number of intersections, hence the number of faces to be found. There are other advantages related to the stability of the numerical implementation that will be discussed in section 8.

We write down below the pivoting equations for  $G_d$ . Let  $\{\xi\}$  be a vertex of  $G_d$  and  $\lambda = \lambda(\xi)$ . This means that there are  $s + d$  active constraints such that

$$(10) \quad \mathbf{C}_{\text{act}} = \begin{bmatrix} -e_{i_1} & a_1 \\ \vdots & \vdots \\ -e_{i_d} & a_d \\ 0 & Q_{d+1} \\ \vdots & \vdots \\ 0 & Q_n \end{bmatrix} \quad \text{and} \quad \mathbf{b}_{\text{act}} = \begin{bmatrix} b(i_1, a_1) \\ \vdots \\ b(i_d, a_d) \\ R^{n-d} r_{d+1} \\ \vdots \\ R r_n \end{bmatrix}$$

are respectively the matrix of active constraints and the vector of active constraints. There are at least  $m_i^{(d-1)} + 1$  occurrences of  $e_i$ . There is a unique  $1 \leq q \leq s$  so that there are  $m_q^{(d-1)} + 2$  occurrences of  $e_q$ , and those are the active constraints that may be ‘dropped’. Facets are uniquely defined by the set of active constraints:

*Lemma 21.* *Let  $R \geq 0$ . Assume that*

$$\mathbf{C}_{\text{act}} \begin{bmatrix} \lambda \\ \xi \end{bmatrix} = \mathbf{b}_{\text{act}}.$$

*Then  $\mathbf{C}_{\text{act}}$  is invertible. Moreover, there cannot be any extra  $(i, a)$  so that*

$$[-e_i, \quad a] \begin{bmatrix} \lambda \\ \xi \end{bmatrix} = \mathbf{b}(i, \mathbf{a})$$

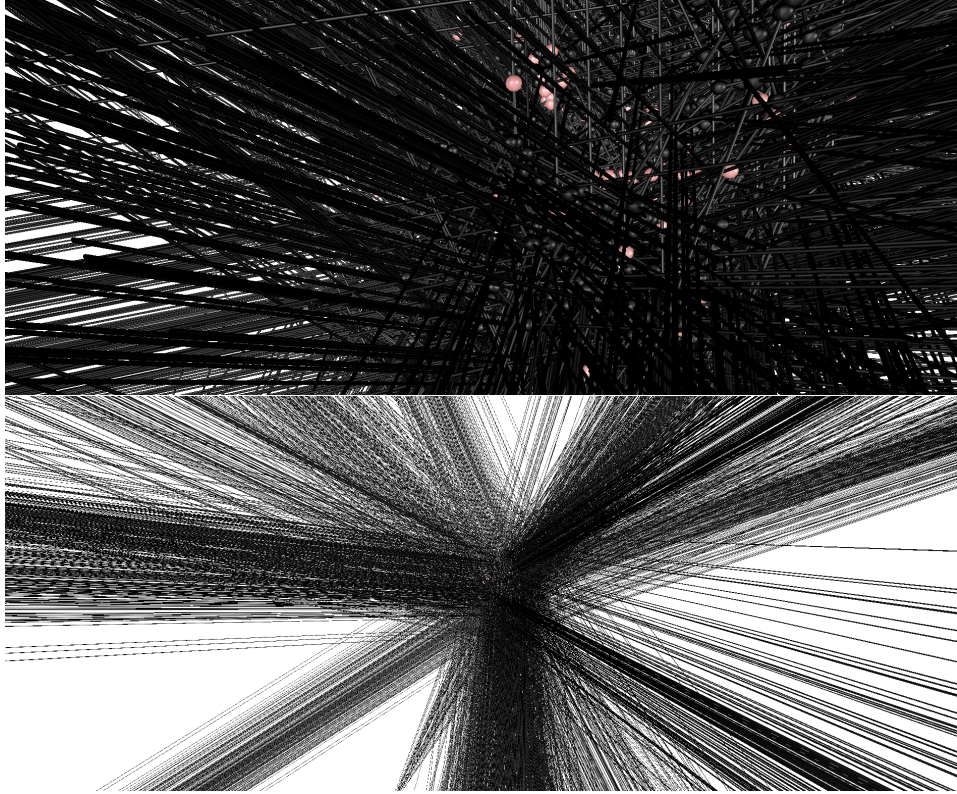


FIGURE 5. Those pictures illustrate the reason we prefer to take an affine flag ‘at infinity’. The top picture shows  $G_{18}$  for the sonic3 polynomial system (18 equations in 18 variables) viewed from close to the origin. The pipes that are interrupted are actually crossing the screen. The bottom picture shows  $G_{18}$  viewed from far away.

*Proof.* The matrix  $\mathbf{C}_{\text{act}}$  is of the form

$$\mathbf{C}_{\text{act}} = \begin{bmatrix} \mathbf{e}_{i_1} & \mathbf{a}_1 \\ \vdots & \vdots \\ \mathbf{e}_{i_d} & \mathbf{a}_d \\ 0 & Q_{d+1} \\ \vdots & \vdots \\ 0 & Q_n \end{bmatrix}$$

Because the flag  $F_0 \subset \cdots \subset F_n$  is generic and the  $A_i$  are finite, the span of  $Q_{d+1}, \dots, Q_n$  is transversal to any space spanned by exactly  $d$  vectors of the form  $a - a'$ , where  $a, a' \in \cup A_i$ . After elimination and some row permutations, matrix

$\mathbf{C}_{\text{act}}$  factors:

$$\mathbf{C}_{\text{act}} = PL \begin{bmatrix} -I & U_{12} \\ & U_{22} \\ & Q_{d+1} \\ & \vdots \\ & Q_n \end{bmatrix}.$$

$P$  is a permutation.  $L$  is lower triangular with  $L_{ii} = 1$ . The rows of  $U_{12}$  are elements of  $\cup A_i$ , and the rows of  $U_{22}$  are of the form  $a - a'$ ,  $a, a' \in A_i$  for the same  $i$ .

Thus,

$$\begin{aligned} \text{rank}(\mathbf{C}_{\text{act}}) &= \text{rank} \left( \begin{bmatrix} \mathbf{e}_{i_1} & \mathbf{a}_1 \\ \vdots & \vdots \\ \mathbf{e}_{i_d} & \mathbf{a}_d \end{bmatrix} \right) + n - d \\ &= \text{rank} \left( \begin{bmatrix} \mathbf{e}_{i_1} & \mathbf{a}_1 & b(i_1, \mathbf{a}_1) \\ \vdots & \vdots & \\ \mathbf{e}_{i_d} & \mathbf{a}_d & b(i_d, \mathbf{a}_d) \end{bmatrix} \right) + n - d. \end{aligned}$$

Should the matrix  $\mathbf{C}_{\text{act}}$  be singular, there will be  $d$  linearly dependent vectors of the form  $[\mathbf{e}_i, \mathbf{a}, b(i, \mathbf{a})] \in \mathbb{R}^{n+s+1}$  contradicting the genericity of the lifting  $b$ .

The same argument shows that should there be another active constraint  $(i, \mathbf{a})$ , there would be  $d+1 \leq n+s_1$  linearly dependent vectors of the form  $[\mathbf{e}_i, \mathbf{a}, b(i, \mathbf{a})] \in \mathbb{R}^{n+s+1}$ , contradiction again.  $\square$

The flag  $F = F_0 \subset \cdots \subset F_n$  will be assumed to be an affine flag ‘at infinity’. This means that the equations for  $F_d$  are of the form

$$\begin{aligned} Q_{d+1}\xi &= R^{n-d}r_{d+1} \\ &\vdots \\ Q_n\xi &= Rr_n \end{aligned}$$

with generic  $Q_i$ , non-zero  $r_i$  and  $R \rightarrow +\infty$ . The values of  $\xi$  and  $\lambda(\xi)$  are now polynomials in  $R$ . Let  $B = \mathbf{C}_{\text{act}}^{-1}$ . The ansatz below is the key to represent those polynomials by their constant term:

$$(11) \quad \begin{bmatrix} \lambda \\ \xi \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} + B \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{n-d}r_{d+1} \\ \vdots \\ Rr_n \end{bmatrix} \quad \text{with} \quad \mathbf{C}_{\text{act}} \begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} = \mathbf{b}_{\text{act}} = \begin{bmatrix} b(i_1, a_1) \\ \vdots \\ b(i_d, a_d) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Only  $\lambda_0$  and  $\xi_0$  need to be stored in memory, the rest of the polynomial is implicit once we know  $\mathbf{C}_{\text{act}}$  and  $B$ .

Inactive constraints  $(i, a)$  (for  $R \rightarrow \infty$ ) satisfy

$$(12) \quad \begin{aligned} &R^{n-d}r_{d+1} [-\mathbf{e}_i, a] B \mathbf{e}_{s+d+1} + \cdots + Rr_n [-\mathbf{e}_i, a] B \mathbf{e}_{n+s} + \\ &+ [-\mathbf{e}_i, a] B \begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} - b(i, a) < 0 \end{aligned}$$



The expression above may be interpreted as a polynomial in  $R$  or, as  $R \rightarrow \infty$  as a non-standard number. It is strictly negative if and only if the higher-order non-zero coefficient is strictly negative. The reader can check that only the sign of the  $r_j$ 's matters:

*Lemma 22.* Assume that  $r_j > 0$  for all  $j$ . If (12) is strictly negative for  $R$  large enough, then for any other choice of the  $r_j > 0$  it will remain negative for  $R$  large enough.

Now we consider the effect of ‘dropping’ the  $j$ -th constraint. As before, we set

$$\begin{bmatrix} \Delta_j \lambda \\ \Delta_j \xi \end{bmatrix} = -B e_j.$$

By Lemma 16, the corresponding edge is of the form

$$\{t \Delta_j \xi, t \in (0, I_j)\}$$

where  $I_j$  can be determined as in Lemma 18. We will need the polynomial

$$t(i, a)(R) = \sum_{l=0}^{n-d} t_l(i, a) R^l$$

where

$$t_0(i, \mathbf{a}) = \frac{a(\xi_0)_i - b(i, a) - (\lambda_0)_i}{[-e_i, \mathbf{a}] B e_j}$$

and for  $l = 1, \dots, n-d$ ,

$$t_l(i, \mathbf{a}) = \frac{[-e_i, \mathbf{a}] B e_{n+s+1-l}}{[-e_i, \mathbf{a}] B e_j} r_{n+1-l}.$$

Let  $\mathcal{C}$  be the set of inactive constraints with  $t(i, a)(R) > 0$  once  $R$  is large enough. There may be inactive constraints with  $[-e_i, \mathbf{a}] B e_j = 0$  but those are not eligible as elements of  $\mathcal{C}$ . Assuming  $\mathcal{C}$  not empty,  $\operatorname{argmin}_{\mathcal{C}} t(i, a)(R)$  denotes the constraint  $(i, a) \in \mathcal{C}$  that is minimal, once  $R$  is large enough.

*Lemma 23.* Let  $\xi \in G_d$  be a vertex. Assume all the notations above. If  $\mathcal{C}$  is not empty, Let  $(i^*, a^*) = \operatorname{argmin}_{(i, a) \in \mathcal{C}} t(i, a)(R)$ ,  $t^* = t(i^*, a^*) = \sum_{l=0}^{n-d} t_l^* R^l$ . Then,

- (a)  $(i^*, a^*)$  is uniquely defined.
- (b) If  $R$  is large enough,  $[\xi, \xi']$  is a segment of  $G_d$ , where  $\xi' = \xi + t^* \Delta \xi$ .
- (c) Let  $\lambda' = \lambda + t^* \Delta \lambda$ . Let  $B'$  be the inverse to the matrix of active constraints at  $\xi'$ . Then

$$\begin{bmatrix} \lambda' \\ \xi' \end{bmatrix} = \begin{bmatrix} \lambda'_0 \\ \xi'_0 \end{bmatrix} + B' \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{n-d} r_{d+1} \\ \vdots \\ R r_n \end{bmatrix}$$

with

$$\begin{bmatrix} \lambda'_0 \\ \xi'_0 \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} - t_0^* B e_j.$$

*Proof.* We prove item (b) before unicity. Let  $0 < \tau < t^*$ . For active constraints  $(k, a_k)$  except the  $j$ -th one (that we ‘dropped’),  $-\tau[-e_k, a_k]Be_j = -\tau\delta_{jk} = 0$  so that

$$a_k(\xi + \tau\Delta_j\xi) - b(k, a) = \lambda_k + \tau\Delta_j\lambda_k.$$

We claim that all the other constraints  $(k, a')$  satisfy

$$a'(\xi + \tau\Delta_j\xi) - b(k, a') < \lambda_k + \tau\Delta_j\lambda_k.$$

For the  $j$ -th constraint, this follows from  $-\tau[-e_j, a_j]Be_j = -\tau < 0$ . Similarly, if  $(k, a')$  is an inactive constraint not in  $C$ , either  $-\tau[-e_k, a']Be_j = 0$  or  $[-e_k, a']Be_j > 0$  and  $a_k\xi - b(k, a^*) - \lambda_k$  have different sign. Since the latter is negative,  $[-e_k, a']Be_j > 0$  and  $-\tau[-e_k, a']Be_j \leq 0$ .

Now assume  $(k, a') \in C$ : since the numerators of  $t(i^*, a^*) \leq t(k, a_k)$  are all negative, we have the inequality  $[e_{i^*}, a^*]Be_j \geq \tau[-e_k, a_k]Be_j$ .

$$\begin{aligned} a_k(\xi + \tau\Delta_j\xi) - b(k, a) - \lambda_k + \tau\Delta_j\lambda_k &= a_k\xi - b(k, a) - \lambda_k - \tau[e_k, a]Be_j \\ &= (t(k, a) - \tau)[e_k, a]Be_j \\ &\leq (t(k, a) - \tau)[e_{i^*}, a^*]Be_j \\ &\leq 0 \end{aligned}$$

with equality if  $(k, a) = (i^*, a^*)$ .

We can prove item (a) now. Should unicity fail, there will be  $n + d + 1$  active constraints for  $\xi + t^*\Delta_j\xi$ , which contradicts Lemma 21. Therefore, the minimum of  $w$  is attained in a unique  $(i^*, a^*) \in C$ .

Let  $\mathbf{C}'_{\text{act}}$  be the matrix of active constraints for  $\xi'$  and let  $B'$  be its inverse. In order to prove item (c), we will first check that for  $1 \leq l < n - d$ ,

$$B'e_{n+s+1-l}r_{n-l+1} = Be_{n+s+1-l}r_{n-l+1} - t_l^*Be_j.$$

Notice that  $\mathbf{C}'_{\text{act}} = \mathbf{C}_{\text{act}} - e_jv$ , with  $v = [-e_j, a_j] - [-e_{i^*}, a^*]$ . Also, let  $\mathbf{b}'_{\text{act}} = \mathbf{b}_{\text{act}} + e_j(b(i^*, a^*) - b(j, a_j))$ . By the previous item and by construction,

$$\mathbf{C}'_{\text{act}} \begin{bmatrix} \lambda + t\Delta_j\lambda \\ \xi + t\Delta_j\xi \end{bmatrix} = \mathbf{b}'_{\text{act}}$$

The invertibility of  $\mathbf{C}'_{\text{act}}$  follows from Lemma 21. Now we apply Lemma 20 to obtain an expression for  $B'$ :

$$B' = B + \frac{1}{1 - v^TB e_j} Be_j v^T B.$$

Note that  $v^TB = e_j^T - [-e_{i^*}, a^*]B$ . Thus, for each  $1 \leq l \leq n - d$ ,

$$\begin{aligned} r_{n+s-l}B'e_{n+s+1-l} &= r_{n+s-l}Be_{n+s+1-l} \\ &\quad + r_{n+s-l} \frac{[-e_j, a_j]Be_{n+s+1-l} - [-e_{i^*}, a^*]Be_{n+s+1-l}}{[-e_{i^*}, a^*]Be_j} Be_j \\ &= r_{n+s-l}Be_{n+s+1-l} - t_l^*Be_j \end{aligned}$$

Hence,

$$B' \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{n-d}r_{d+1} \\ \vdots \\ Rr_n \end{bmatrix} = B \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{n-d}r_{d+1} \\ \vdots \\ Rr_n \end{bmatrix} - \sum_{l=1}^{n-d} R^l t_l B e_j$$

Thus,

$$\begin{aligned} \begin{bmatrix} \lambda' \\ \xi' \end{bmatrix} &= \begin{bmatrix} \lambda \\ \xi \end{bmatrix} - t^* B e_j \\ &= \begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} + B \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{n-d}r_{d+1} \\ \vdots \\ Rr_n \end{bmatrix} - t^* B e_j \\ &= \begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} - t_0^* B e_j \end{aligned}$$

□

Lemma 23 allows us to explore each of the sets  $G_d$ ,  $1 \leq d \leq n$  and hence to produce  $X_d$ , as long as we have at least one vertex from each connected component of  $G_d$ . Those vertices can be recovered from  $X_{d-1}$  by using Theorem 13 and the lemma below.

The following Lemma allows to find starting points in  $G_{d+1}$  by exploring  $X_d$ . In order to do that, we ‘drop’ the  $s + d + 1$ -th constraint. Let  $w$  and  $C$  be defined accordingly:

$$t(i, a)(R) = \sum_{l=0}^{n-d} t_l(i, a) R^l$$

where

$$t_0(i, \mathbf{a}) = \frac{a(\xi_0)_i - b(i, a) - (\lambda_0)_i}{[-e_i, \mathbf{a}] B e_{s+d+1}}$$

and for  $l = 1, \dots, n - d$ ,

$$t_l(i, \mathbf{a}) = \frac{[-e_i, \mathbf{a}] B e_{n+s+1-l}}{[-e_i, \mathbf{a}] B e_{s+d+1}} r_{n+1-l}.$$

Let  $\mathcal{C}$  be the set of inactive constraints with  $t(i, a)(R) \neq 0$  once  $R$  is large enough. It is important to notice that for  $R$  large, item (d) of Theorem 13 reads:

(d') All points in  $G_d \cap F_{d-1}$  are in the interior of a half line of  $G_d$ .

Therefore, all constraints in  $\mathcal{C}$  have value of the same sign (positive or negative).

*Lemma 24. With the notations of Lemma 23, let  $j = s + d + 1$ . Let  $\xi \in X_d$  be a vertex. If  $C$  is not empty, Let  $(i^*, a^*) = \operatorname{argmin}_{(i,a) \in C} |t(i, a)(R)|$ ,  $t^* = t(i^*, a^*) = \sum_{l=0}^{n-d} t_l^* R^l$ . Then,*

(a)  $(i^*, a^*)$  is uniquely defined.

- (b) If  $R$  is large enough,  $[\xi, \xi')$  is a half line of  $G_d$ , where  $\xi' = \xi + t^* \Delta \xi$ . Moreover,  $\xi'$  is a point of  $G_{d+1}$ .
- (c) Let  $\lambda' = \lambda + t^* \Delta \lambda$ . Let  $B'$  be the inverse to the matrix of active constraints at  $\xi'$ . Then

$$\begin{bmatrix} \lambda' \\ \xi' \end{bmatrix} = \begin{bmatrix} \lambda'_0 \\ \xi'_0 \end{bmatrix} + B' \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{n-d-1} r_{d+2} \\ \vdots \\ R r_n \end{bmatrix}$$

with

$$\begin{bmatrix} \lambda'_0 \\ \xi'_0 \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} - t_0^* B e_j.$$

*Proof.* The proof of items (a) and (b) is similar to the proof of Lemma (23). Therefore we will only prove item (c).

Let  $\mathbf{C}'_{\text{act}}$  be the matrix of active constraints for  $\xi'$  and let  $B'$  be its inverse. As before,  $\mathbf{C}'_{\text{act}} = \mathbf{C}_{\text{act}} - e_j v$  where  $v = [0, Q_{d+1}] - [-e_{i^*}, a^*]$  is the difference between the  $j$ -th row of  $\mathbf{C}_{\text{act}}$  and the  $j$ -th row of  $\mathbf{C}'_{\text{act}}$ , and  $\mathbf{b}'_{\text{act}} = e_{d+1} b(i^*, a^*)$ . We have

$$\mathbf{C}'_{\text{act}} \begin{bmatrix} \lambda + t \Delta_j \lambda \\ \xi + t \Delta_j \xi \end{bmatrix} = \mathbf{b}'_{\text{act}}$$

The invertibility of  $\mathbf{C}'_{\text{act}}$  follows from Lemma 21, and

$$B' = B + \frac{1}{1 - v^T B e_{s+d+1}} B e_{s+d+1} v^T B$$

with  $v^T B = e_{s+d+1}^T - [-e_{i^*} a^*] B$ . As in Lemma 23 item (c), for  $1 \leq l < n - d$ ,

$$B' e_{n+s-l+1} r_{n-l+1} = B e_{n+s-l+1} r_{n-l+1} - t_l^* B e_{s+d+1}.$$

When  $l = n - d$ ,  $t_l^* = r_{d+1}$ . Therefore  $R^{n-d} r_{d+1} B e_{s+d+1} - R^{n-d} t_{n-d}^* B e_{s+d+1} = 0$  and

$$\begin{aligned} \begin{bmatrix} \lambda' \\ \xi' \end{bmatrix} &= \begin{bmatrix} \lambda \\ \xi \end{bmatrix} - t^* B e_j \\ &= \begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} + B \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{n-d} r_{d+1} \\ \vdots \\ R r_n \end{bmatrix} - t^* B e_j \\ &= \begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} - t_0^* B e_j \end{aligned}$$

□

There is a reciprocal to Lemma 24.

*Lemma 25.* Let  $\xi'$  be a vertex from  $G_{d+1}$ . Let  $q$  be the unique integer such that  $m_q(\xi') > m_q^{(d)}$ . Then there is  $\xi \in X_d$  such that  $[\xi, \xi']$  is a half line of  $G_d$  if and only if the following conditions hold for some active constraint  $(q, a)$  of  $\xi'$  (say the  $j$ -th):

- (a) The set  $\mathcal{C}$  corresponding to ‘dropping’ constraint  $j$  is empty.
- (b)  $[0, Q_{d+1}]B'e_j < 0$ .

In that case, with the notations of Lemma 24,

$$\begin{bmatrix} \lambda \\ \xi \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} + B \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{n-d}r_{d+1} \\ \vdots \\ Rr_n \end{bmatrix}$$

with

$$\begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} = \begin{bmatrix} \lambda'_0 \\ \xi'_0 \end{bmatrix} + \frac{r_{d+1}}{[0, Q_{d+1}]B'e_j} B'e_j.$$

## 5. THE MAIN ALGORITHM

Here is a simplified version of the algorithm. Let  $\mathcal{V}_d$  denote the set of points (vertices) of the tropical curve  $G_d$  and  $\mathcal{E}_d$  the set of finite segments (edges) in  $G_d$ ,  $1 \leq d \leq n$ . The degree of the graph  $\Gamma_d = (\mathcal{V}_d, \mathcal{E}_d)$  is at most  $\max m_i^{(d-1)}$ . Consider now the graph  $\Gamma = (\mathcal{V}, \mathcal{E})$  where  $\mathcal{V} = \cup \mathcal{V}_d$  and  $\mathcal{E}$  is the union of all the  $\mathcal{E}_d$  with the set of connecting segments  $[\xi, \xi']$  from Lemma 24,  $\xi \in \mathcal{V}_d$ ,  $\xi' \in \mathcal{V}_{d+1}$ . The degree of  $\Gamma$  is at most  $1 + 2 \max m_i^{(d-1)}$ . The AllMixedCells algorithm is a graph walk through  $\Gamma$ .

The algorithm will operate on two sets: a set  $\mathcal{V}_{\text{pending}} \subset \mathcal{V}$  to explore, and a set  $\mathcal{V}_{\text{known}} \subset \mathcal{V}$  of already explored lower faces. Notice that  $\#\mathcal{V} \leq \sum_{d=2}^n v_d$  where  $v_d$  is the number of vertices of  $G_d$ . Therefore  $\#\mathcal{E} \leq (\sum_{d=2}^n v_d) \deg(\Gamma)/2$ .

The operator  $\text{Visited}(\mathcal{V}_{\text{known}}, L)$  returns true if  $L \in \mathcal{V}_{\text{known}}$ . Otherwise, it inserts  $L$  in  $\mathcal{V}_{\text{known}}$  and returns false. The simplified graph exploration algorithm is:

**Algorithm ALLMIXEDCELLS**

$\mathcal{V}_{\text{pending}} \leftarrow \{L_{F_0}\}.$

$\mathcal{V}_{\text{known}} \leftarrow \emptyset$

**While**  $\mathcal{V}_{\text{pending}} \neq \emptyset,$

    Remove some  $L_\xi$  from  $\mathcal{V}_{\text{pending}}.$

**If not**  $\text{VISITED}(\mathcal{V}_{\text{known}}, L_\xi)$

**If**  $L_\xi$  is a mixed cell, **then** output  $L_\xi.$

**For** each neighbor  $L_{\xi'}$  of  $L_\xi$  in  $\Gamma$ , insert  $L_{\xi'}$  in  $\mathcal{V}_{\text{pending}}.$

    Discard  $L_\xi.$

We assume that elements of  $\sqcup A_i$  are uniquely labeled by an integer from 1 to  $\sum \#A_i$ . A face  $L$  in  $\mathcal{V}_{\text{pending}}$  is represented by its list of active constraints, and each active constraint is represented by an integer. Thus, a face  $L_\xi$  for  $\xi \in \mathcal{V}_d$  is represented by a strictly increasing list of  $s + d$  integers. From this representation one can find the inverse  $B$  to the matrix of active constraints in time  $O(n^\omega)$  with

$\omega < 2.3728639$  the exponent of matrix multiplication (Vassilevska Williams, 2012; Le Gall, 2014). Once we obtained  $B$  it is easy to compute  $\xi_0$  and  $\lambda_0$ .

The neighbors  $L_{\xi'}$  can be listed by the formulas in Lemmas 23 to 25. Not all inactive constraints need to be tested. Suppose we drop the  $j$ -th constraint. All other active constraints will be called the *remaining* constraints.

*Lemma 26. Let  $(i, \mathbf{a})$ ,  $(i, \mathbf{a}')$  be active constraints in the same lower face. Then,*

$$[(\mathbf{a}, b(i, \mathbf{a})), (\mathbf{a}', b(i, \mathbf{a}'))]$$

*is a sharp lower edge of*

$$\hat{A}_i = \text{Conv}(\{(\mathbf{a}, b(i, a)) : a \in A_i\}) \subset \mathbb{R}^n \times \mathbb{R}.$$

It follows from the Lemma that the only inactive constraints to check are those  $(i^*, a^*)$  so that  $[(\mathbf{a}^*, b(i^*, \mathbf{a}^*)), (\mathbf{a}', b(i^*, \mathbf{a}'))]$  is a sharp lower edge of  $\hat{A}_{i^*}$  for all remaining constraints  $\mathbf{a}'$ . Thus,

*Lemma 27. Assume that a lower face  $L_\xi$  is given and that the matrix  $B$  inverse to the matrix of active constraints is known.*

- (1) *If  $s = 1$  and  $d = n$ , the neighboring lower face of  $L_\xi$  can be produced in time at most*

$$O(n^2 \#A_1).$$

- (2) *If  $n = s$  and  $m_1 = \dots = m_n = 2$  and the sharp lower edges of the  $\hat{A}_i$ 's are known, then the neighboring lower faces of  $L_\xi$  can be produced in time at most*

$$O(n^2 \sum (E_i)).$$

*where  $E_i$  is the maximal number of lower sharp edges incident to a sharp vertex of  $\hat{A}_i$ .*

*Proof.* If  $s = 1$ , there are  $n + 1$  active constraints that may be 'dropped' by using the formulas in Lemmas 23 to 25. Since  $d = n$ , testing an inactive constraint costs  $O(n)$  arithmetic operations. Testing all inactive constraints, the argmin in the formulas costs at most  $n^2 \#A_1$ .

In the second case, there are two or three constraints that may be 'dropped'. Testing an inactive constraint costs  $O(n(n + 1 - d)) \leq O(n^2)$  arithmetic operations. There are at most  $\sum E_i$  inactive constraints to test for the argmin. □

*Remark 28.* One does not need to explore edges going from  $\mathcal{V}_d$  to  $\mathcal{V}_{d-1}$ .

*Remark 29.* If  $L_\xi$  and  $L_{\xi'}$  are adjacent and  $B$  is known, the inverse  $B'$  of the matrix of active constraints of  $L_{\xi'}$  can be produced by a rank-one update with cost  $O(n^2)$ .

In order to bound the cost of each WHILE iteration inside algorithm ALLMIXED-CELLS, we still need to bound the cost of storing and retrieving data from sets  $\mathcal{V}_{\text{pending}}$  and  $\mathcal{V}_{\text{known}}$ . We represented each lower face  $L$  by a list of active constraints, that is a unique strictly increasing list of integers  $(l_1, \dots, l_{s+d})$  from 1 to  $\sum \#A_i$ . To each of those lists, we associate a *hash value*, that is a real number in  $[0, 1]$  computed by the formula:

$$\sigma(L) = H_1 l_1 + \dots + H_{s+d} l_{s+d} \mod 1.$$

where the  $H_j \in [0, 1]$  are uniformly distributed random numbers computed once and for all before starting ALLMIXEDCELLS. The probability of a *collision*, that is

of two different lower faces having the same value of  $\sigma$ , is zero. The *hash function*  $\sigma$  will be used to order the lower faces in such a way that the comparison cost is one.

The sets  $\mathcal{V}_{\text{pending}}$  and  $\mathcal{V}_{\text{known}}$  are represented by balanced trees (Knuth, 1998, Ch.6). For lower faces in  $\mathcal{V}_{\text{known}}$ , only the hash value needs to be stored. The cost of the operator VISITED is therefore bounded above by  $O(\log(\#\mathcal{V}))$ . For the case of  $\mathcal{V}_{\text{pending}}$ , we can also store the list of active constraints. Thus:

*Lemma 30. Under the assumptions above, the cost of storing or retrieving a value in  $\mathcal{V}_{\text{pending}}$  or  $\mathcal{V}_{\text{known}}$  is at most  $O(n + \log(\#\mathcal{V}))$ .*

The algorithm below supports the bounds in Theorem 1. It proceeds in two steps. First it computes the sharp lower edges of each  $\hat{A}_i$  (See Lemma 27). For each  $\mathbf{a} \in A_i$ , its lifting  $(\mathbf{a}, b_i(\mathbf{a}))$  is a sharp lower vertex of  $\hat{A}_i$  if and only if it belongs to the border of a sharp lower edge. We will now replace the lifting  $b_i : A_i \rightarrow [0, 1]$  with a new general lifting,  $\tilde{b}_i : A_i \rightarrow [0, 2]$  with

$$\tilde{b}_i(\mathbf{a}) = \begin{cases} b_i(\mathbf{a}) & \text{if } (\mathbf{a}, b_i(\mathbf{a})) \text{ a sharp lower vertex, and} \\ b_i(\mathbf{a}) + 1 & \text{otherwise.} \end{cases}$$

The second step is the ALLMIXEDCELLS algorithm applied to the lifting  $(\tilde{b}_1, \dots, \tilde{b}_n)$ .

**Algorithm** ALLMIXEDCELLSFULL( $n, A_1, \dots, A_n$ )  
**Draw** a random lifting  $b : \sqcup_i A_i \rightarrow [0, 1]$ , each coordinate uniformly distributed.  
**Draw** uniform random numbers  $H_1, \dots, H_{2n}$  uniformly distributed in  $[0, 1]$ .  
**Draw** an orthogonal matrix  $Q$  uniformly distributed with respect to the Haar measure of  $SO(n)$  and define  $F_i$  as the space orthogonal to the last  $n - i$  columns of  $Q$ .  
**For**  $i = 1, \dots, n$  invoke ALLMIXEDCELLS with  $s = 1$  and input  $A_i, b_i$ . Produce a list of the lower edges of  $\hat{A}_i$ .  
**Set**  $V \leftarrow 0$ .  
 Invoke ALLMIXEDCELLS with  $s = n, m_i = 2$  and input  $A_1, \dots, A_n, \tilde{b}_1, \dots, \tilde{b}_n$ .  
 For each mixed lower face, add the volume of the mixed cell to  $V$  and output the lower face.  
**Output**  $V$ .

We can prove the first part of Theorem 1:

*Lemma 31. Let  $d_i = \dim \text{Conv}(A_i)$  and  $V_i = d_i! \text{Vol}(\text{Conv}(A_i))_{d_i}$ . Let*

$$T = \left( \sum_{i=2}^d v_i \right) \left( n^2 \sum_{i=1}^n E_i + \log \sum_{i=2}^n v_i \right), T' = (\max V_i) \left( n^2 \sum_{i=1}^n \#A_i + \log \max_{i=1, \dots, n} V_i \right).$$

*The algorithm ALLMIXEDCELLSFULL performs at most  $O(T + T')$  arithmetic operations.*

*Proof.* Uniform random numbers  $b(i, a)$  and  $H_i$  can be obtained at unit cost. To produce the matrix  $Q \in SO(n)$ , one first produces a random matrix  $A$ , where each coordinate  $A_{ij}$  is an independently distributed  $N(0, 1)$  random numbers. By using standard algorithms like the Box-Müller method, one obtains each  $A_{ij}$  within bounded cost. Then,  $Q$  may be produced by computing the QR factorization of  $A$ , or equivalently be Gram-Schmidt orthonormalization of the columns of  $A$ .

Computing the dimension  $d_i$  of each  $A_i$  is easy. There are at most

$$d_i! \text{Vol}(\text{Conv}(A_i))_{d_i}$$

lower faces to explore, and the first one can be found by standard linear programming techniques. The cost of each exploration step is bounded by Lemma 27(1). Therefore, the total cost is  $O(T')$ .

The last call to ALLMIXEDCELLS must explore  $\#\mathcal{V} = \sum v_i$  lower faces, and the cost of exploring each one was bounded in Lemma 27(2). The total cost of this step is therefore  $O(T)$ .  $\square$

*Remark 32.* The Box-Müller method requires the use of a logarithm, a square root, sine and cosine. The Gram-Schmidt method or QR factorization uses square roots. Rather than approximating those functions, I assumed that they are available at unique cost.

## 6. DETERMINISTIC COMPLEXITY ANALYSIS

Let  $v_d$  be the number of vertices of  $G_d$ . The number  $v_d$  depends on the lifting  $b : A_1 \sqcup \dots \sqcup A_s \rightarrow \mathbb{R}$  that is assumed generic, but fixed. It also depends on the flag  $(F_0 \subset \dots \subset F_n)$  that we take of the form

$$F_{n-d}(R) = \{\xi \in \mathbb{R}^n : Q_{d+1}\xi = R^{n-d}, \dots, Q_{n-1}\xi = R^2, Q_n\xi = R\}$$

where  $Q_1, \dots, Q_n$  are the rows of a generic matrix in  $O(n)$ , and  $R > 0$  is assumed to be large enough.

Let  $A_0 = \{0, e_1, -e_1, e_2, \dots, -e_n\}$  and  $\mathcal{A}_0 = \text{Conv}(A_0)$ . The solid  $\mathcal{A}_0$  is also known as the  $n$ -orthoplex and denoted by  $\beta_n$ . Alternative terminologies are  $n$ -cross and *cocube*. Notice that

$$\frac{1}{\sqrt{n}}B^n \subset \beta_n \subset B^n.$$

The main result in this section is:

*Proposition 33.*

$$v_d \leq n!V(\mathcal{A}_1, m_1^{(d-1)}; \dots; \mathcal{A}_s, m_s^{(d-1)}; \mathcal{A}, 1; \beta_n, n-d).$$

Towards the proof of Proposition 33 we extend the lifting  $b$  to  $A_0 = \beta_n$  by

$$b(0, 0) = 0 \quad \text{and} \quad b(0, \pm e_i) = S$$

where  $S > 0$  will be determined later.

*Definition 34.* Let  $W \subset A_0 \sqcup \dots \sqcup A_s$ . The set of viable points for  $W$  is

$$\begin{aligned} \Xi(W) = \{ \xi \in \mathbb{R}^n : \forall (i, a) \in A_0 \sqcup \dots \sqcup A_s, \\ a\xi - \lambda_i(\xi) - b_i(a) \leq 0 \\ \text{with equality iff } (i, a) \in W \}. \end{aligned}$$



To every  $W \subset A_0 \sqcup \cdots \sqcup A_s$  we associate the pair

$$(13) \quad \mathbf{C}_{\text{act}} = \mathbf{C}_{\text{act}}(W) = \begin{bmatrix} \vdots & \vdots \\ -\mathbf{e}_i & a \\ \vdots & \vdots \\ 0 & Q_{e+1} \\ \vdots & \vdots \\ 0 & Q_n \end{bmatrix} \quad \text{and} \quad \mathbf{b}_{\text{act}} = \mathbf{b}_{\text{act}}(W) = \begin{bmatrix} \vdots \\ b_i(a) \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where  $e = \#W - (s+1)$  and  $(i, a) \in W$ . We say that  $W$  is linearly independent if  $\mathbf{C}_{\text{act}}(W)$  is invertible. In that case we also set  $B = B(W) = \mathbf{C}_{\text{act}}(W)^{-1}$ .

*Proof of Proposition 33.* We start by fixing  $S$ . To that end, we notice that for every linearly independent set  $W$  such that

$$\exists S, R(S) > 0 \text{ s.t. } \forall R > R(S), \Xi(W) \cap F_e(R) \neq \emptyset,$$

equation (11) determines  $\xi(R) \in \Xi(W) \cap F_e(R)$  as

$$\begin{bmatrix} \lambda(R) \\ \xi(R) \end{bmatrix} = B \mathbf{b}_{\text{act}} + B \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{n-e} \\ \vdots \\ R \end{bmatrix}.$$

The first term is a constant when  $W \cap A_0 = \{0\}$ , otherwise it is affine in  $S$ . The second term is a polynomial in  $R$ .

For any  $(i, a) \notin W$ , write

$$t(i, a) = [\mathbf{e}_i \ a] \begin{bmatrix} \lambda(R) \\ \xi(R) \end{bmatrix} - b_i(a) = f_{W,i,a}(R) + g_{W,i,a}S$$

where  $f_{W,i,a}$  is a polynomial in  $R$  and  $g_{W,i,a}$  is a constant, vanishing when  $W \cap A_0 = \{0\}$ . By hypothesis, the expression above is negative for  $R$  large enough. This means that the leading term of  $f_{W,i,a}$  is strictly negative.

As there are finitely many  $W, i$  and  $a$ , there exists a uniform constant  $R_d > 0$  such that when  $R \geq R_d$ , each of the  $f_{W,i,a}(R)$  for all  $d \leq e \leq n-1$  is strictly negative and non-increasing. This constant  $R_d$  is independent of  $S$ . Now pick  $S$  large enough so that if  $W \cap A_0 = \{0\}$ ,  $\|\xi(R_d)\|_\infty < S$ . Notice that if  $(0, 0) \in W$  and  $(0, \pm \mathbf{e}_i) \in W$  then forcefully  $\xi_i = \pm S$ .

For each  $d \leq e \leq n$ , let  $\mathcal{W}_e$  be the class of all subsets  $W \subset A_0 \sqcup \cdots \sqcup A_s$  such that

- (a)  $(0, 0) \in W$ .
- (b)  $W$  is linearly independent.
- (c)  $\#W = s + e + 1$ .
- (d)  $\#W \cap A_0 = 1 + e - d$ .
- (e)  $\#W \cap A_i \geq 1 + m_i^{(d-1)}$ , and
- (f) For  $R$  large enough,  $\Xi(W) \cap F_e(R) \neq \emptyset$ . This holds in particular for some  $R$  with  $\|\xi(R)\|_\infty \leq S$ .

**Induction hypothesis in  $e \in \{d, d+1, \dots, n\}$ :** For every vertex  $\xi(R)$  of  $G_d$ , there is one and only one  $W \in \mathcal{W}_e$  with  $\xi(R) \in \Xi(W)$  for  $R > R_e$  for some  $R_e$  with  $\|\xi(R_e)\|_\infty \leq S$ .

**Base step  $e = d$ :** Let  $W = \{(0, 0)\} \cup \{(i, a) \in A_i, i \geq 1 : -\lambda_i(\xi(R)) + a\xi(R) - b(i, a) = 0\}$ . For this choice of  $W$ ,  $g_{W,i,a} = 0$  at all inactive constraints. By construction of  $G_d$ ,  $W \in \mathcal{W}_d$  and  $\xi(R) \in \Xi(W)$ .

**Induction step:** Let  $W \in \mathcal{W}_e$ ,  $d < e < n$ . Assume after reshuffling indexes and changing signs that  $W \cap A_0 = \{0, e_1, \dots, e_{e-d}\}$ . For some value of  $R_{e+1} > R_e$ , the curve  $(\xi_{e-d+1}(R), \dots, \xi_n(R))$  will exit the hypercube  $\max_{i > e-d}(|\xi_i|) = S$ . Say this happens for  $\xi_{e-d+1}(R_{e+1})$ . Then we set  $W' = W \cup \{(0, e_{e-d+1})\}$ .

The point  $\xi(R_{e+1})$  belongs to  $\Xi(W')$ . By construction,  $W'$  is linearly independent. So we can construct  $\mathbf{C}_{\text{act}}(W')$  and  $B(W')$ , and hence all the  $f_{W',i,a}$  and  $g_{W',i,a}$  for inactive constraints  $(i, a) \notin W'$ . With  $R \geq R_{e+1}$ , set:

$$\begin{bmatrix} \lambda'(R) \\ \xi'(R) \end{bmatrix} = B(W') \mathbf{b}_{\text{act}}(W') + B(W') \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{n-e-1} \\ \vdots \\ R \end{bmatrix}.$$

Then for all inactive constraints,

$$-[\mathbf{e}_i, a] \begin{bmatrix} \lambda'(R) \\ \xi'(R) \end{bmatrix} - b(a) = f_{W',i,a}(R) + g_{W',i,a}(S).$$

Since  $R \geq R_{e+1} > R_e$ ,  $f_{W',i,a}(R') < 0$  is negative and non-increasing. Thus

$$f_{W',i,a}(R) + g_{W',i,a}(S) \leq f_{W',i,a}(R_{e+1}) + g_{W',i,a}(S) < 0.$$

It follows that for  $R \geq R_{e+1}$ ,  $\Xi(W') \cap F_e(R') \neq \emptyset$ . Moreover,  $\|\xi'(R_{e+1})\|_\infty = S$ .

**Conclusion.** By induction, we can associate injectively to each vertex of  $G_d$ , an element of  $\mathcal{W}_n$  which is a mixed cell for one  $(A_0, n-d; A_1, m_1^{(d-1)} + \delta_{1p}; \dots; A_s, m_s^{(d-1)} + \delta_{sp})$ . The volume of such a mixed cell is an integral multiple of  $1/n!$ . Proposition 33 follows.  $\square$

## 7. AVERAGE COMPLEXITY ANALYSIS

Proposition 33 holds for a generic lifting  $b$  and for a sufficiently generic flag  $F_0 \subset \dots \subset F_n$ . Now we assume that the orthogonal group  $O(n)$  is endowed with the Haar probability measure. For each  $Q \in O(n)$ , let  $Q_d$  be the  $d$ -th row of  $Q$ . Let  $R$  be an arbitrarily large positive real number. As before, the flag of affine subspaces is:  $F_n = \mathbb{R}^n$ ,

$$F_{d-1} = \{\xi \in F_d : Q_d \xi = R^{n-d+1}\}.$$

Then we set  $\bar{v}_d = \text{Avg}_{Q \in O(n)} v_d$ . The following result should be compared with Proposition 33:

*Proposition 35. Suppose that  $V(\mathcal{A}_1, m_1; \dots; \mathcal{A}_s, m_s) \neq 0$ . Then,*

$$\bar{v}_d \leq \frac{n!}{2^{n-d}} V(\mathcal{A}_1, m_1^{(d-1)}; \dots; \mathcal{A}_s, m_s^{(d-1)}; \mathcal{A}, 1; B^n, n-d).$$

We will use the following fact to establish Proposition 35:

*Lemma 36.* Suppose that the topological closure  $\overline{\Xi(W)}$  of  $\Xi(W)$  contains a line. Then  $V(\mathcal{A}_1, m_1; \dots; \mathcal{A}_s, m_s) = 0$ .

*Proof.* Let the line be  $\xi_0 + t\dot{\xi}$ . Without loss of generality, assume that  $a\dot{\xi} = 0$  for all active constraints  $(i, a) \in W$ . For inactive constraints,

$$-\lambda_i + a\xi_0 + ta\dot{\xi} \leq b(i, a) \quad \forall t.$$

This is only possible if  $a\dot{\xi} = 0$ . So  $A_1, \dots, A_s \subset \dot{\xi}^\perp$ .  $\square$

*Proof of Proposition 35.* The case  $d = n$  follows from Proposition 33 so we assume  $d < n$ . Let  $\mu_1 + \dots + \mu_s = d$ . We define  $\mathcal{W}(\mu_1, \dots, \mu_s)$  as the class of subsets  $W \in A_1 \sqcup \dots \sqcup A_s$  with  $\#W \cap A_i = \mu_i + 1$  and such that  $\Xi(W)$  contains an affine cone of dimension  $n - d$ .

Every vertex of  $G_d$  corresponds to such a subset for some choice of  $\mu_i \geq m_i^{(d-1)}$ . Indeed, the curve  $\xi(R)$  obtained by varying  $R$  large enough cannot be contained in an hyperplane of codimension  $d + 1$ . So what we need to do is to count the average number of elements of  $\mathcal{W}(\mu_1, \dots, \mu_s)$ .

Let  $\mathbf{C}_{\text{act}}(W, Q)$  be as in (13) for a particular choice of  $Q$ . Then,

$$\max_{Q \in O(n)} \det(\mathbf{C}_{\text{act}}(W, Q)) = \max_{Q_{d+1}, \dots, Q_n \in B^n} \det(\mathbf{C}_{\text{act}}(W, Q)) \geq 1.$$

Also, let  $\mathcal{W}(\mu_1, \dots, \mu_s, Q)$  be the set of  $W \in \mathcal{W}(\mu_1, \dots, \mu_s)$  such that  $\Xi(W) \cap F_d(R)$  is not empty, for  $R$  sufficiently large. Then,

$$\begin{aligned} \text{Avg}_{Q \in O(n)} \# \mathcal{W}(\mu_1, \dots, \mu_s; Q) &\leq \\ &\leq \sum_{W \in \mathcal{W}(\mu_1, \dots, \mu_s)} \text{Prob}[\Xi(W) \cap F_{n-d}(R) \neq \emptyset] \\ &\leq \sum_{W \in \mathcal{W}(\mu_1, \dots, \mu_s)} \left( \text{Prob}[\Xi(W) \cap F_{n-d}(R) \neq \emptyset] \max_{Q_{d+1}, \dots, Q_n \in B^n} \det(\mathbf{C}_{\text{act}}(W, Q)) \right) \\ &\leq \left( \max_W \text{Prob}[\Xi(W) \cap F_{n-d}(R) \neq \emptyset] \right) \left( \sum_W \max_{Q_{d+1}, \dots, Q_n \in B^n} \det \mathbf{C}_{\text{act}}(W, Q) \right) \end{aligned}$$

The second term is bounded above by

$$n! V(\mathcal{A}_1, \mu_1; \dots; \mathcal{A}_s, \mu_s; B^n, n - d).$$

We claim now that

$$\text{Prob}[\Xi(W) \cap F_{n-d}(R) \neq \emptyset] \leq \frac{1}{2^{n-d}}.$$

Indeed, let  $W \in \mathcal{W}(\mu_1, \dots, \mu_s)$  and admit that there is  $Q$  such that

$$\xi(R) \in \Xi(W) \cap F_{n-d}(R)$$

exists for  $R$  large enough. Let

$$I_n \neq \Sigma = \begin{bmatrix} I_d & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{bmatrix}$$

be a non-trivial sign change matrix, and let  $Q' = \Sigma'Q$ . Let  $F'_{n-d}(R)$  be the induced flag. Suppose that  $\xi'(R) \in \Xi(W) \cap F'_{n-d}(R)$ .

Since  $\Xi(W)$  is a convex, the line segment  $[\xi(R), \xi'(R)]$  is contained in  $\Xi(W)$ . Making  $R \rightarrow \infty$ , we conclude that  $\overline{\Xi(W)}$  contains a straight line. Then Lemma 36 implies that the mixed volume of the  $\mathcal{A}_i$  vanishes, contradiction. Therefore,  $\Xi(W) \cap F'_{n-d}(R) = \emptyset$ .

The multiplicative group of all the sign matrices  $\Sigma$  as above preserves the Haar metric in  $O(n)$ , and has order  $2^{n-d}$ . Since only one of the  $\Sigma Q$  can have  $\Xi(W) \cap F_{n-d}(R) \neq \emptyset$ , we deduce that

$$\text{Prob}[\Xi(W) \cap F_{n-d}(R) \neq \emptyset] \leq \frac{1}{2^{n-d}}.$$

□

## 8. IMPLEMENTATION NOTES

In this section, I describe several techniques that were used in the software and have an effect on the experiments.

**Hermite normal form.** The polytopes  $A_1, \dots, A_n$  are translated so that each one has a vertex equal to zero. Then, the program computes a basis for the lattice generated by  $\cup(A_i)$  through a Hermite normal form factorization. The supports  $A_i$  are then rewritten into lattice basis coordinates. This may reduce the mixed volume in families of examples such as *Cyclic- $n$*  or *Gridanti- $n$* . If the mixed volume algorithm is coupled to a path-following algorithm, this technique reduces the number of paths to track.

**Hash function.** In Section 5, we associated a unique real number to each possible lower face. To a lower face  $L_\xi$  with active constraints labeled by integers  $l_1 < \dots < l_{s+d}$  we associated the hash value

$$\sigma(L_\xi) = \sum_{j=1}^{s+d} l_j H_j \mod 1$$

with  $H_j$  independently distributed random numbers in  $[0, 1]$ . In the program, those are replaced by pseudo-random floating point numbers. Since pseudo-random numbers are a good proxy for irrational numbers, we expect the values of  $\sigma$  to be well spread from each other (Knuth, 1998, Th.S Sec. 6.4). The reason for choosing the  $l_j$  in increasing order is to ensure that the floating point value associated to a face  $L_\xi$  is always the same.

**Parallelization.** If  $N$  processors are available, the  $k$ -th *node* or processor ( $0 \leq k < N$ ) is in charge of lower faces  $L_\xi$  for  $k \leq N\sigma(L_\xi) < k+1$ . The sets  $\mathcal{V}_{\text{pending}}$  and  $\mathcal{V}_{\text{known}}$  are distributed between the processors: each processor stores the lower faces in its range as a balanced tree.

Below is a crude version of the parallel algorithm, running on node  $k$  out of  $N$ . The new parallel operations are explained afterwards.

**Algorithm** ALLMIXEDCELLS (Parallel)

**If**  $k \leq N\sigma(L_{F_0}) < k + 1$   
     **then**  $\mathcal{V}_{\text{pending}} \leftarrow \{L_{F_0}\}$ .  
     **else**  $\mathcal{V}_{\text{pending}} \leftarrow \emptyset$ .  
 $\mathcal{V}_{\text{known}} \leftarrow \emptyset$   
 $\text{work\_to\_do} \leftarrow \text{True}$ .  
**While**  $\text{work\_to\_do}$   
      $L_\xi \leftarrow \emptyset$ .  
     **While**  $\mathcal{V}_{\text{pending}} \neq \emptyset$  **and**  $L_\xi = \emptyset$ ,  
         Remove some  $L_\xi$  from  $\mathcal{V}_{\text{pending}}$ .  
         **If** VISITED( $\mathcal{V}_{\text{known}}, L_\xi$ ) **then**  $L_\xi = \emptyset$ .  
     **If**  $L_\xi \neq \emptyset$ ,  
         **If**  $L_\xi$  is a mixed cell, **then** output  $L_\xi$ .  
         **For** each neighbor  $L_{\xi'}$  of  $L_\xi$  in  $\Gamma$ , **send**  $L_{\xi'}$  to processor  $\lfloor \sigma(L_{\xi'})/N \rfloor$ .  
         Discard  $L_\xi$ .  
     **Wait** until all sent messages are available to the recipient node.  
     **Receive** all  $L_{\xi'}$ 's sent to node  $k$  and insert them in  $\mathcal{V}_{\text{pending}}$ .  
      $\text{work\_to\_do} \leftarrow (\mathcal{V}_{\text{pending}} \neq \emptyset)$   
     **Reduce** (  $\text{work\_to\_do}$ , **or**,  $0 \leq k < N$ )

Parallel machines communicate by sending messages between processors. Operations **send** and **receive** refer to a message from a given processor, sent to a specified processor. Each node can check whether a message went through, that is whether it is available to the recipient. It can check for incoming messages.

The **reduce** operation (modeled on MPI\_ALLREDUCE) takes three arguments, *variable*, *operation* and *range*. It allows to efficiently compute an aggregated value out of a variable at each node in the range, for the given operation. In the example above, the variables *work* at each processor are ‘or-ed’, and the result is propagated to all the nodes in the range.

**The choice of the function  $m_i^{(d)}$ .** The choice of the  $m_i^{(d)}$  makes a difference. I opted to reorder the  $A_i$ ’s by increasing dimension, then increasing volume, then increasing number of points. In the unmixed case,  $m_i^{(d)} = 2$  for  $i \leq d$  and  $m_i^{(d)} = 1$  when  $i > d$ .

**Numerical stability.** Numerical stability is an issue. Instead of computing rank-1 updates, I opted for producing the matrix  $B$  independently for each lower face. Then I stipulated a value of  $\epsilon = 10^5 \epsilon_M$  where  $\epsilon_M$  is the ‘machine epsilon’ for double precision. The value of  $B$  is always assumed correct, in the sense that

$$\|B_{\text{computed}} - B_{\text{true}}\|_\infty \leq \epsilon \|B_{\text{computed}}\|_\infty.$$

The bounds for the condition number provided by the Lapack library are not always correct. Therefore, I estimated the condition number of  $\mathbf{C}_{\text{act}}$  by  $\|\mathbf{C}_{\text{act}}\|_\infty \|B_{\text{computed}}\|_\infty$ . This was used to bound the forward error  $\|B_{\text{computed}} - B_{\text{true}}\|_\infty$ .

**Recovery step:** When necessary, the precision of the matrix  $B$  can be improved by Newton iteration, where the residual  $\mathbf{C}_{\text{act}} B - I$  is computed using long double or quadruple precision using Newton iteration (Demmel, 1997, Sec.2.5). This is a very rare event.

**Nonstandard numbers:** Numerators and denominators for each of the  $t_l(i, a)$  can be computed with absolute error no more than  $2(1 + \max d_i)(n + s)\epsilon$ . If the computed absolute value of the numerator (resp. denominator) of  $t_l(i, a)$  is below that bound,

Example	$n$	$\sum E_i$	Visited faces	T	AVG $RT$	Std dev	$RT/T$
Cyclic13	13	133	$3.07 \times 10^{06}$	$7.51 \times 10^{10}$	$2.06 \times 10^{02}$	$7.73 \times 10^{00}$	$2.74 \times 10^{-09}$
Cyclic14	14	157	$1.10 \times 10^{07}$	$3.66 \times 10^{11}$	$8.50 \times 10^{02}$	$2.79 \times 10^{01}$	$2.33 \times 10^{-09}$
Cyclic15	15	183	$4.40 \times 10^{07}$	$1.96 \times 10^{12}$	$4.07 \times 10^{03}$	$1.22 \times 10^{02}$	$2.08 \times 10^{-09}$
Noon18	18	324	$8.10 \times 10^{06}$	$8.92 \times 10^{11}$	$1.23 \times 10^{03}$	$1.42 \times 10^{02}$	$1.38 \times 10^{-09}$
Noon19	19	361	$1.64 \times 10^{07}$	$2.24 \times 10^{12}$	$2.87 \times 10^{03}$	$4.08 \times 10^{02}$	$1.28 \times 10^{-09}$
Noon20	20	400	$3.37 \times 10^{07}$	$5.62 \times 10^{12}$	$6.46 \times 10^{03}$	$6.53 \times 10^{02}$	$1.15 \times 10^{-09}$
Chandra18	18	307	$3.32 \times 10^{06}$	$3.45 \times 10^{11}$	$5.18 \times 10^{02}$	$3.63 \times 10^{01}$	$1.50 \times 10^{-09}$
Chandra19	19	343	$7.17 \times 10^{06}$	$9.27 \times 10^{11}$	$1.27 \times 10^{03}$	$5.44 \times 10^{01}$	$1.37 \times 10^{-09}$
Chandra20	20	381	$1.57 \times 10^{07}$	$2.50 \times 10^{12}$	$3.08 \times 10^{03}$	$1.96 \times 10^{02}$	$1.23 \times 10^{-09}$
Chandra21	21	421	$3.46 \times 10^{07}$	$6.67 \times 10^{12}$	$7.58 \times 10^{03}$	$4.59 \times 10^{02}$	$1.14 \times 10^{-09}$
Katsura15	16	200	$5.03 \times 10^{06}$	$2.73 \times 10^{11}$	$5.57 \times 10^{02}$	$6.15 \times 10^{01}$	$2.04 \times 10^{-09}$
Katsura16	17	225	$1.41 \times 10^{07}$	$9.72 \times 10^{11}$	$1.88 \times 10^{03}$	$2.69 \times 10^{02}$	$1.93 \times 10^{-09}$
Katsura17	18	252	$3.55 \times 10^{07}$	$3.05 \times 10^{12}$	$5.31 \times 10^{03}$	$5.57 \times 10^{02}$	$1.74 \times 10^{-09}$
Katsura18	19	280	$8.30 \times 10^{07}$	$8.81 \times 10^{12}$	$1.42 \times 10^{04}$	$9.02 \times 10^{02}$	$1.61 \times 10^{-09}$
Gaukwa7	14	98	$9.78 \times 10^{05}$	$2.01 \times 10^{10}$	$7.00 \times 10^{01}$	$4.83 \times 10^{00}$	$3.48 \times 10^{-09}$
Gaukwa8	16	128	$1.07 \times 10^{07}$	$3.73 \times 10^{11}$	$1.02 \times 10^{03}$	$9.30 \times 10^{01}$	$2.73 \times 10^{-09}$
Vortex5	10	120	$1.37 \times 10^{05}$	$1.83 \times 10^{09}$	$7.62 \times 10^{00}$	$6.47 \times 10^{-01}$	$4.16 \times 10^{-09}$
Vortex6	15	240	$7.53 \times 10^{07}$	$4.39 \times 10^{12}$	$7.25 \times 10^{03}$	$9.17 \times 10^{02}$	$1.65 \times 10^{-09}$
N-body5	10	130	$2.94 \times 10^{05}$	$4.30 \times 10^{09}$	$1.42 \times 10^{01}$	$1.78 \times 10^{00}$	$3.31 \times 10^{-09}$
Gridanti3	18	81	$7.10 \times 10^{04}$	$1.93 \times 10^{09}$	$9.26 \times 10^{00}$	$4.28 \times 10^{-01}$	$4.80 \times 10^{-09}$
Gridanti4	32	144	$3.08 \times 10^{07}$	$4.62 \times 10^{12}$	$1.14 \times 10^{04}$	$5.98 \times 10^{02}$	$2.47 \times 10^{-09}$
Sonic8	28	407	$6.78 \times 10^{06}$	$2.21 \times 10^{12}$	$2.49 \times 10^{03}$	$4.37 \times 10^{02}$	$1.13 \times 10^{-09}$
Sonic9	30	452	$1.38 \times 10^{07}$	$5.70 \times 10^{12}$	$5.89 \times 10^{03}$	$9.25 \times 10^{02}$	$1.03 \times 10^{-09}$
Sonic10	32	497	$2.34 \times 10^{07}$	$1.21 \times 10^{13}$	$1.18 \times 10^{04}$	$1.56 \times 10^{03}$	$9.73 \times 10^{-10}$
Graphmodel6	21	48	$3.37 \times 10^{04}$	$7.29 \times 10^{08}$	$6.50 \times 10^{00}$	$6.65 \times 10^{-01}$	$8.92 \times 10^{-09}$
Graphmodel7	28	63	$4.14 \times 10^{05}$	$2.08 \times 10^{10}$	$1.11 \times 10^{02}$	$7.52 \times 10^{00}$	$5.33 \times 10^{-09}$
Graphmodel8	36	80	$7.43 \times 10^{06}$	$7.80 \times 10^{11}$	$3.85 \times 10^{03}$	$3.38 \times 10^{02}$	$4.94 \times 10^{-09}$
Eco20	20	209	$1.11 \times 10^{07}$	$9.61 \times 10^{11}$	$1.93 \times 10^{03}$	$5.25 \times 10^{02}$	$2.00 \times 10^{-09}$
Eco21	21	230	$2.36 \times 10^{07}$	$2.48 \times 10^{12}$	$4.62 \times 10^{03}$	$1.30 \times 10^{03}$	$1.86 \times 10^{-09}$
Reimer13	13	169	$6.99 \times 10^{03}$	$2.10 \times 10^{08}$	$1.54 \times 10^{00}$	$1.22 \times 10^{-01}$	$7.33 \times 10^{-09}$
Reimer14	14	196	$1.15 \times 10^{04}$	$4.62 \times 10^{08}$	$2.18 \times 10^{00}$	$2.51 \times 10^{-01}$	$4.72 \times 10^{-09}$
Reimer15	15	225	$1.92 \times 10^{04}$	$1.02 \times 10^{09}$	$3.35 \times 10^{00}$	$2.20 \times 10^{-01}$	$3.29 \times 10^{-09}$
VortexAC4	12	84	$1.17 \times 10^{04}$	$1.50 \times 10^{08}$	$1.55 \times 10^{00}$	$6.69 \times 10^{-02}$	$1.03 \times 10^{-08}$
VortexAC5	20	200	$3.19 \times 10^{06}$	$2.64 \times 10^{11}$	$4.84 \times 10^{02}$	$5.09 \times 10^{01}$	$1.83 \times 10^{-09}$

TABLE 1. Predicted value of  $T$  and measured running time for selected examples (8 cores).

it is assumed to be zero. Similarly, if  $|t_l(i, a) - t_l(i', a')| < \epsilon \|[-e_i a]Be_j\|^{-1}$ , those quantities are deemed equal.

**Random walk:** I experimented with a random walk strategy to find all the connected components of  $G_n$  in  $\Gamma$ , instead of the deterministic walk through all of  $\Gamma$ . Each connected component found was explored by the *AllMixedCells* algorithm. Then other random paths were explored until an heuristic stopping criterion was met. The results are discussed on section 9.

## 9. NUMERICAL RESULTS

**9.1. Choice of the examples.** I selected a few families of benchmark systems for which a general formula is available or is easy to figure. In particular, the benchmark families tested by Lee and Li (2011) were all included in the benchmark: *Cyclic*, *Noon*, *Chandra*, *Katsura*, *Gaukwa*, *Vortex*, *N-body*, *Gridanti* and *Sonic*. Precise references for most of them were given by Verschelde (1999).

Number of cores:	8	16	32	64	Efficiency exp.
Cyclic15	$4.07 \times 10^{03}$	$2.06 \times 10^{03}$	$1.15 \times 10^{03}$	$5.90 \times 10^{02}$	$9.20 \times 10^{-01}$
Noon20	$6.46 \times 10^{03}$	$3.25 \times 10^{03}$	$1.81 \times 10^{03}$	$9.00 \times 10^{02}$	$9.37 \times 10^{-01}$
Chandra21	$7.58 \times 10^{03}$	$3.81 \times 10^{03}$	$2.14 \times 10^{03}$	$1.08 \times 10^{03}$	$9.28 \times 10^{-01}$
Gaukwa9	Unavail.	$8.25 \times 10^{03}$	$4.57 \times 10^{03}$	$2.27 \times 10^{03}$	$9.30 \times 10^{-01}$

TABLE 2. Parallel timing. The efficiency exponent is the quadratic best fit for the linear coefficient of  $(\log(N_i), -\log(T_i))$  where  $T_i$  is the measured running time with  $N_i$  cores.

Example	$n$	Mixed volume	det	Output size	AllMixedCells (8 cores)	Random path (8 cores)	Lee and Li (1 core)
Cyclic12	12	500352	12	$5.00 \times 10^{05}$	$3.95 \times 10^{01}$	$1.91 \times 10^{01}$	$5.70 \times 10^{01}$
Cyclic13	13	2704156	13	$2.70 \times 10^{06}$	$2.06 \times 10^{02}$	$1.14 \times 10^{02}$	$5.04 \times 10^{02}$
Cyclic14	14	8795976	14	$8.80 \times 10^{06}$	$8.50 \times 10^{02}$	$4.10 \times 10^{02}$	$4.03 \times 10^{03}$
Cyclic15	15	35243520	15	$3.52 \times 10^{07}$	$4.07 \times 10^{03}$	$1.91 \times 10^{03}$	$3.64 \times 10^{04}$
Cyclic16	16	135555072	16	$1.36 \times 10^{08}$		$8.28 \times 10^{03}$	
Noon18	18	387420453	1	$6.97 \times 10^{09}$	$1.23 \times 10^{03}$		
Noon19	19	1162261429	1	$2.21 \times 10^{10}$	$2.87 \times 10^{03}$		$6.35 \times 10^{02}$
Noon20	20	3486784361	1	$6.97 \times 10^{10}$	$6.46 \times 10^{03}$		$1.11 \times 10^{03}$
Noon21	21	10460353161	1	$2.20 \times 10^{11}$		$3.56 \times 10^{03}$	$4.30 \times 10^{03}$
Noon22	22	31381059565	1	$6.90 \times 10^{11}$		$8.72 \times 10^{03}$	$9.21 \times 10^{03}$
Noon23	23	94143178781	1	$2.17 \times 10^{12}$		$1.98 \times 10^4$	$2.43 \times 10^{04}$
Chandra18	18	131072	1	$2.36 \times 10^{06}$	$5.18 \times 10^{02}$		
Chandra19	19	262144	1	$4.98 \times 10^{06}$	$1.27 \times 10^{03}$		
Chandra20	20	524288	1	$1.05 \times 10^{07}$	$3.08 \times 10^{03}$		$4.62 \times 10^{02}$
Chandra21	21	1048576	1	$2.20 \times 10^{07}$	$7.58 \times 10^{03}$		$1.07 \times 10^{03}$
Chandra22	22	2097152	1	$4.61 \times 10^{07}$			$2.60 \times 10^{03}$
Chandra23	23	4194304	1	$9.65 \times 10^{07}$			$7.38 \times 10^{03}$
Chandra24	24	8388608	1	$2.01 \times 10^{08}$			$1.80 \times 10^{04}$
Katsura13	14	8190	1	$1.15 \times 10^{05}$			$1.32 \times 10^{02}$
Kastura14	15	16254	1	$2.44 \times 10^{05}$			$6.02 \times 10^{02}$
Katsura15	16	32730	1	$5.24 \times 10^{05}$	$5.57 \times 10^{02}$	$1.69 \times 10^{01}$	$2.57 \times 10^{03}$
Katsura16	17	65280	1	$1.11 \times 10^{06}$	$1.88 \times 10^{03}$	$2.85 \times 10^{01}$	$1.46 \times 10^{04}$
Katsura17	18	131070	1	$2.36 \times 10^{06}$	$5.31 \times 10^{03}$	$4.68 \times 10^{01}$	$7.56 \times 10^{04}$
Katsura18	19	261576	1	$4.97 \times 10^{06}$	$1.42 \times 10^{04}$	$9.97 \times 10^{01}$	
Katsura19	20	524286	1	$1.05 \times 10^{07}$		$1.67 \times 10^{02}$	
Kastura20	21	1047540	1	$2.20 \times 10^{07}$		$3.68 \times 10^{02}$	
Gaukwa7	14	11390625	1	$1.59 \times 10^{08}$	$7.00 \times 10^{01}$	$9.14 \times 10^{00}$	$2.75 \times 10^{02}$
Gaukwa8	16	410338673	1	$6.57 \times 10^{09}$	$1.02 \times 10^{03}$	$7.98 \times 10^{01}$	$1.07 \times 10^{04}$
Gaukwa9	18	16983563041	1	$3.06 \times 10^{11}$		$1.00 \times 10^{03}$	$3.70 \times 10^{05}$
Gaukwa10	20	794280046581	1	$1.59 \times 10^{13}$		$1.28 \times 10^{04}$	
Eco19	19	131072	1	$2.49 \times 10^{06}$	$9.28 \times 10^{02}$		
Eco20	20	262144	1	$5.24 \times 10^{06}$	$1.93 \times 10^{03}$	$1.21 \times 10^{02}$	
Eco21	21	524288	1	$1.10 \times 10^{07}$	$4.62 \times 10^{03}$	$2.39 \times 10^{02}$	
Eco22	22	1048576	1	$2.31 \times 10^{07}$	$8.75 \times 10^{03}$	$4.75 \times 10^{02}$	
Eco23	23	2097152	1	$4.82 \times 10^{07}$		$1.06 \times 10^{03}$	

TABLE 3. Measured output-sensitivity of the full ALLMIXED-CELLS algorithm, of ALLMIXEDCELLS with the random path heuristic and results reported by Lee and Li (2011).

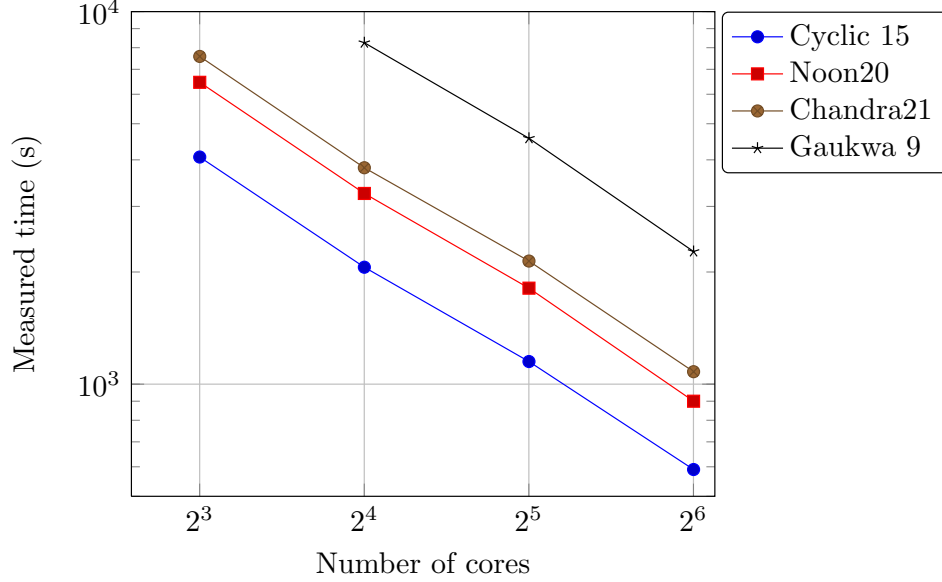


FIGURE 6. Measured running time (using 8 cores) against the invariant  $T$  from equation (2) for several benchmark examples. Data from table 1 page 30.

The system *VortexAC* from (Chen et al., 2014) was also included for the sake of comparison. Due to hardware limitations, I was unable to compute the mixed volume of *VortexAC*-6.

Jan Verschelde maintains a list of polynomial systems in <http://homepages.math.uic.edu/~jan/demo.html>. From his list I selected the *Eco* and the *Reimer* families.

According to Morgan (1987, p.148(7.3)), the system *Eco*- $n$  arises from economic modeling. It is defined by

$$\begin{aligned}
 -c_i + x_i x_n + \sum_{j=1}^{n-i-1} x_j x_{i+j} x_n &= 0 & \text{for } i = 1, \dots, n-1, \\
 x_1 + \dots + x_{n-1} + 1 &= 0
 \end{aligned}$$

The *Reimer*- $n$  family was defined in the Posso suite, still available at <http://www-sop.inria.fr/saga/POL/>. It is given by:

$$-1/2 + \sum_{j=1}^n (-1)^{j+1} x_j^{i+1} = 0 \quad \text{for } 1 \leq i \leq n.$$

The family *Graphmodel* comes from Gaussian graphical models in statistics (Uhler, 2012). Let  $G$  be the cyclic graph with vertices  $\{1, \dots, n\}$  and edges  $\mathcal{E} = \{\{1, 1\}, \{2, 2\}, \dots, \{n, n\} \cup \{1, 2\}, \{2, 3\}, \dots, \{n, 1\}\}$ . Let  $X$  and  $Y$  be  $n \times n$  symmetric matrices, constructed as follows. Assume that  $i \leq j$ : If  $\{i, j\} \in \mathcal{E}$ , then  $X_{ij}$  is a variable and  $Y_{ij}$  is a random complex number. If  $\{i, j\} \notin \mathcal{E}$ , then  $Y_{ij}$  is a variable and  $X_{ij} = 0$ .



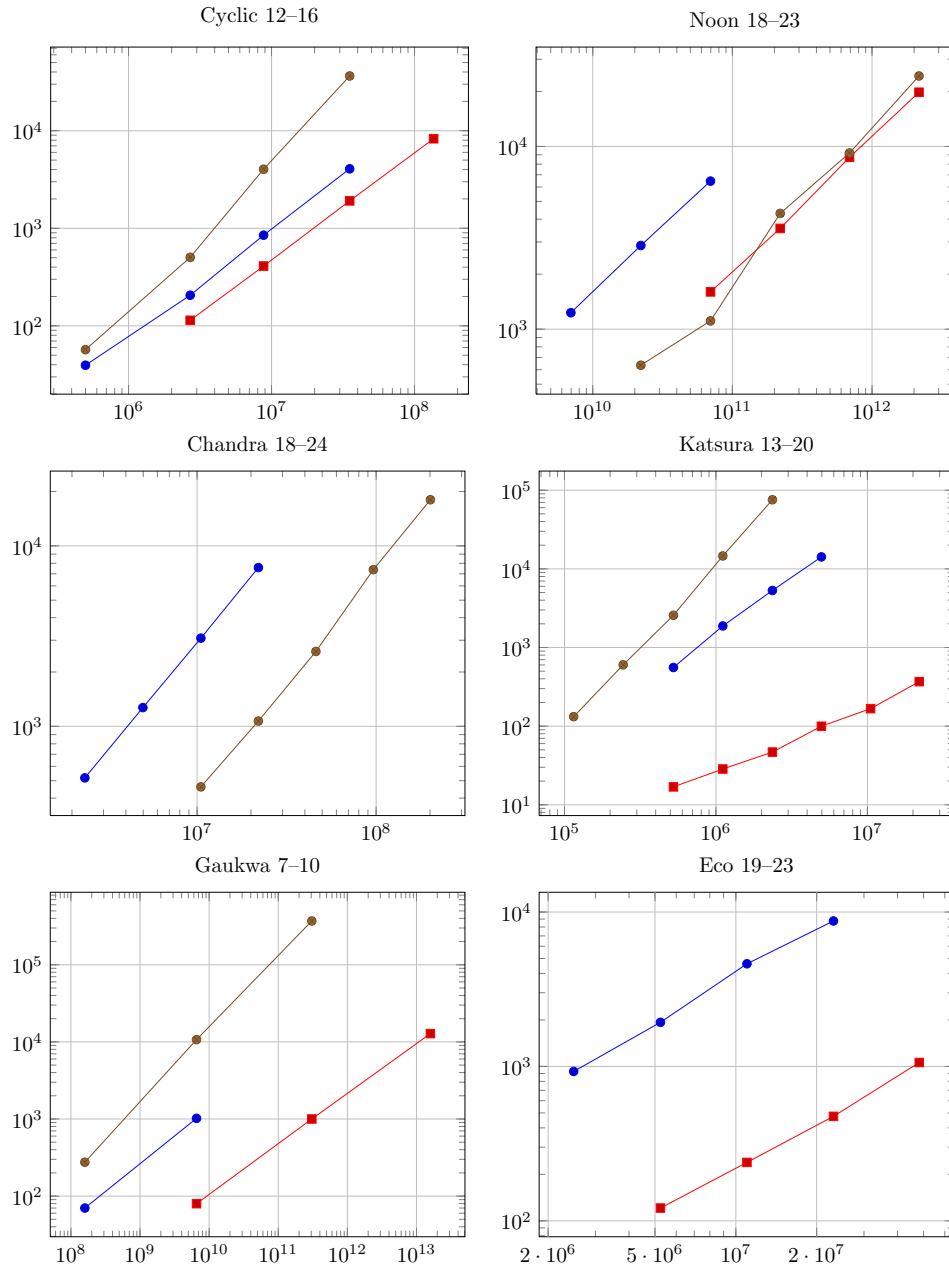


FIGURE 7. Measured running time of ALLMIXEDCELLS ( $\bullet$ ), ALLMIXEDCELLS with random path ( $\blacksquare$ ) and running time published by Lee and Li (2011) ( $\bullet$ ) against the output size for selected examples. All times in seconds. Data from table 3 page 31.

The system *Graphmodel-n* is given by the upper triangular part of the matrix equation  $XY = I$ . The actual number of roots of the overdetermined system  $XY = I$  is known as the *Maximum Likelihood degree* of the graphical model  $G$ . The same construction may be carried out for any graph  $G$ .

**9.2. Hardware.** Computations were performed at NACAD (*Núcleo Avançado de Computação de Alto Desempenho*) in the *Universidade Federal do Rio de Janeiro*. The machine used was a SGI Altix ICE 8400 running Intel MKL (includes Lapack) and MVAPICH2 (MPI implementation). I used up to 8 nodes, with 8 cores per node. The CPUs are either Six Core Intel Xeon X5650 (Westmere) running at 2.67 GHz or Quad Core Intel Xeon X5355 (Clovertown) running at 2.66 GHz.

**9.3. Claim 1: adequacy of the computation model.** Each of the benchmark examples was tested for 10 pseudo-random liftings. The results are displayed in table 1. The number of visited faces is an average. The column  $T$  displays the bound in (2). The running time is the average wall time for all steps of the program, from reading the input to writing the mixed cells to the output file. For systems large enough, the running time was found to be of the order of  $10^{-9}T$  seconds, with a standard deviation under 15%.

**9.4. Claim 2: scaling.** Four benchmark examples with  $T > 10^{12}$  were selected for the scaling test (Table 2).

In order to evaluate the efficiency of parallelization, I used least squares to compute the linear coefficient of the best affine approximation for data  $(\log N_i, -\log T_i)$  where  $N_i$  is the number of cores and  $T_i$  the average running time. I obtained a running time of

$$O(N^{0.93})$$

while perfect, linear parallelization would yield  $O(N)$ .

**9.5. Claim 3: comparison with other available software.** The best published timings for finding mixed cells are those in (Lee and Li, 2011) and (Chen et al., 2014). Since experiments were performed in different machines, the absolute timings may not be comparable. However, the time ratio from a benchmark example to the next example in the same family is an invariant.

To make sense from this invariant, I plotted the running time against the output size, in a log-log scale (Fig. 7).

The slopes of the lines show how the running time increases with respect to the output size. A slope close to one or smaller implies that computing the mixed cells will not be a bottleneck for the overall polynomial solving by homotopy.

Fig. 7 shows that there is not a best algorithm for all cases and that all the three tested algorithms are competitive for some of the benchmark families. It is possible to see an asymptotic gain in running time for the families *Cyclic*, *Katsura* and *Gaukwa* when using ALLMIXEDCELLSFULL.

Comparison with the results by (Chen et al., 2014) can only be done in terms of absolute running time, adjusting their results in the shared memory model to 8 cores. Their program had a similar running time for the *Cyclic-15* and *Eco-20* examples and was faster than ALLMIXEDCELLSFULL for *Sonic-8* and *Katsura-15*. However the number of cores using shared memory is limited so those results are not necessarily scalable.

## 10. CONCLUSIONS

We introduced a new algorithm to compute mixed cells and mixed volumes. Its running time was bounded in terms of quermassintegrals associated to the supporting polytopes of the equations. This is the first non-combinatorial bound for mixed volume computation.

The implementation of the algorithm is competitive with available software. Its main drawbacks are memory usage and some numerical stability problems for very large polynomial systems.

Memory usage problems disappear when using a sufficient number of processors, since most memory storage is local and distributed.

Numerical instability arises when two faces are nearly parallel, or when the matrix of active constraints is nearly degenerate. This problem was solved through rigorous error bounds and judicious use of quadruple precision arithmetic. Extra precision may be required if the number of faces to visit becomes substantially larger than in the tested examples (table 1). At this time precision is not an issue, so this is left for future implementations.

The random walk method for accelerating the algorithm is a promising strategy. It should be coupled with a fast mixed volume estimator (unavailable at this time) to ensure correctness of the results. Moreover, random graph search algorithms are a research subject by itself.

While the motivation of this paper was to provide good starting systems for homotopy, the numerical implementation of polyhedral homotopy continuation may require adequate mathematical machinery beyond projective spaces and unitary group action. See for instance (Malajovich and Rojas, 2004; Malajovich, 2013) on conditioning and root counting on toric varieties. Homotopy algorithms on toric varieties will be the subject of a future paper.

*Note.* While this paper was under review, Jensen (2016) proposed a symbolic algorithm for tropical homotopy continuation, using similar but subtly different ideas. Preliminary experiments suggest a running time comparable to the random path method (table 3), yet it is deterministic.

## GLOSSARY OF NOTATIONS

$V(\mathcal{A}_1, \dots, \mathcal{A}_n)$	Mixed volume of $\mathcal{A}_1, \dots, \mathcal{A}_n$ .	p.2
$A_i$	Finite subset of $\mathbb{Z}^n$ .	p.2
$V$	Scaled mixed volume $n!V(\text{Conv}(A_1), \dots, \text{Conv}(A_n))$ .	p.2
$V_i$	Generic root bound of an unmixed system of support $A_i$ .	p.3
$T, T'$	Time bounds for the algorithm	p.3
$s$	Number of different supports $A_i$ .	p.6
$m_i$	Multiplicity of each support $A_i$ .	p.6
$b_i = b_i(\mathbf{a}) = b(i, \mathbf{a})$	Lifting value for $\mathbf{a} \in A_i$ .	p.6
$\lambda_i = \lambda_i(\xi)$	Legendre dual for the lifting $b_i$ .	p.6
$m_i(\xi)$	Number of times $\lambda_i(\xi)$ attained, minus one.	p.7
$L_{i,\xi}$	Facet of $\text{Graph}(\hat{b}_i)$ .	p.7
$L_\xi$	Facet of $\text{Graph}(\sum t_i \hat{b}_i)$ , $t_i$ indeterminates.	p.7
$\Xi(L)$	Possibly unbounded polyhedron dual to face $L$ .	p.8
$F_0 \subset \dots \subset F_n$	Generic affine flag in $\mathbb{R}^n$ .	p.8
$m_i^{(d)}$	Certain non-decreasing sequence.	p.8
$X_d$	Certain zero-dimensional tropical variety.	p.8
$G_d$	Certain one-dimensional tropical variety.	p.8
$\mathbf{C}, \mathbf{b}$	Cayley matrix and lifting vector.	p.9
$q$	Last polytope so that $m_q^{(d)}$ increased at time $d$ .	p.11
$\Delta_j$	Pivoting direction, in $\xi$ -space.	p.11
$I_j$	Pivoting distance.	p.11
$\Delta_j \xi, \Delta_j \lambda$	Pivoting vectors while dropping constraint $j$ .	p.11
$\mathbf{C}_{\text{act}}, \mathbf{b}_{\text{act}}$	Matrix and vector of active constraints.	p.11
$\mathbf{C}_{\text{inact}}, \mathbf{b}_{\text{inact}}$	Matrix and vector of inactive constraints.	p.12
$t(i, \mathbf{a})$	score of inactive constraint $[i, \mathbf{a}]$	p.13
$Q_i$	Unit vector orthogonal to $F_{i-1}$ in $F_i$ .	p.16
$R$	non-standard number, $R > k$ for all $k \in \mathbb{R}$ .	p.16
$B$	Inverse to the matrix $\mathbf{C}_{\text{act}}$ of active constraints.	p.16
$t_{i,a}(R)$	Scores for inactive constraints.	p.17
$\Gamma = (\mathcal{V}, \mathcal{E})$	Graph to be explored. Union of tropical curves.	p.21
$v_d$	number of vertices of $G_d$ .	p.21
$E_i$	Degree of 1-skeleton of lifting of $A_i$	p.22

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